PARTIAL DIFFERENTIAL EQUATIONS

Partial Differential Equation :—

The differential equation having differential co efficients w.r. to two or more independent variables is said to be partial differential equation.

Therefore a partial differential equation contains one dependent and two or more independent variables.

In a relation $z=f(x,y)$. x,y are independent variables and z is a dependent variable so that we have to find the partial derivative of z w.r.t x,y.

Eg :- $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

Order and Degree of the partial differential equation :—

The order of a partial differential equation is the order of the highest partial derivative occuring in the equation and its degree is the power of the highest order partial derivative in the equation.

Eg:- $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ is of first order and first degree.

$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x \partial y}=0$ is of second order and first degree.

For a relation $z=f(x,y)$.

The partial derivative of z, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ will be denoted by p and q respectively.

The second partial derivatives $\frac{\partial^2 z}{\partial x^2}$ $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ are denoted by r, t and s respectively.

so that $\frac{\partial^2 z}{\partial x^2} = r$ $\frac{\partial^2 z}{\partial x \partial y} = s$ $\frac{\partial^2 z}{\partial y^2} = t$.

Formation of Partial Differential Equations :—

If the number of arbitrary constants to be eliminated is equal to the number of independent variables involved in the equation, then we obtain a partial differential equation of first order.

11) Find the differential equation of all spheres of radius 5 having their centres in the xy-plane.

Sol:- The equation of the family of spheres having centre at (a,b,c) and radius r are. $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.

The equation of the family of spheres having their centres in the xy-plane and having radius 5 is

$$(x-a)^2 + (y-b)^2 + z^2 = 25 \quad \text{—— (1)}.$$

Diff (1) partially w.r.t x and y, we get

$$2(x-a) + 2z \frac{\partial z}{\partial x} = 0.$$

$$(x-a) = -zp \quad \text{—— (2)}.$$

$$2(y-b) + 2z \frac{\partial z}{\partial y} = 0$$

$$(y-b) = -zq \quad \text{—— (3)}$$

Sub. the values of $(x-a)$ and $(y-b)$ in (1), we get.

$$z^2 p^2 + z^2 q^2 + z^2 = 25.$$

$$z^2 (p^2 + q^2 + 1) = 25$$

This is the required diff. equation.

Form the partial differential equation by eliminating the arbitrary constants a, b from $z = a \log\left[\frac{b(y-1)}{1-x}\right]$

Sol:- Given that $z = a\log\left[\frac{b(y-1)}{(1-x)}\right]$

$$z = a\left[\log b(y-1) - \log(1-x)\right]$$

$$z = a\log b + a\log(y-1) - a\log(1-x) \quad \text{——} (1)$$

The number of arbitrary constants to be eliminated is equal to the number of independent variables involved in the equation then we obtain a partial differential equation of first order.

Diff (1) w.r.t x and y partially, we get

$$\frac{\partial z}{\partial x} = -\frac{a}{1-x}(-1)$$

$$p = \frac{a}{1-x}.$$

$$p(1-x) = a \quad \text{——} (2)$$

$$\frac{\partial z}{\partial y} = \frac{a}{y-1}$$

$$q = \frac{a}{y-1}$$

$$q(y-1) = a \quad \text{——} (3)$$

From (2) and (3), we get

$$p(1-x) = q(y-1)$$

$$px + qy = p + q.$$

This is the required partial differential equation.

Form the diff. equation of all planes having equal intercepts on x and y axes.

Sol:- We know that the equation of the plane having intercepts on x, y and z axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

The equation of the plane having equal intercepts on x and y axes is $\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1$ —— (1)

The number of arbitrary constants to be eliminated is equal the number of independent variables involved in the equation, so we obtain a partial differential equation of first order

Diff (1) w.r.t x and y partially, we get

$$\frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial x} = 0$$

$$\frac{1}{a} = -\frac{p}{c} \quad \text{—— (2)}$$

$$\frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial y} = 0.$$

$$\frac{1}{a} = -\frac{q}{c} \quad \text{—— (3)}$$

From (2) and (3), we get

$$-\frac{p}{c} = \frac{-q}{c}$$

$$p = q.$$

This is the required partial diff. equation.

Note:- If the number of arbitrary constants to be eliminated is more than the number of Independent variables involved in the equation then the partial differential equation is second or higher order.

Eg:- (1) Form the partial differential equation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol:- Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ——(1)

Where a, b, c are arbitrary constants.

Here we have to treat z as a fun. of x and y

The No. of arbitrary constants are three is more than the number of independent variables are two. So we obtain a partial differential eqn of second or higher order.

Diff (1) partially w.r.t 'x', we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\frac{x}{a^2}+\frac{z}{c^2}p=0 \quad \text{——(2)}$$

Diff (1) partially w.r.t 'y', we get

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$$

$$\frac{y}{b^2}+\frac{z}{c^2}q=0 \quad \text{——(3)}$$

Diff (2) partially w.r.t 'y', we get

$$\frac{1}{c^2}\left[z\frac{\partial p}{\partial y}+p\frac{\partial z}{\partial y}\right]=0$$

$$z\frac{\partial p}{\partial y}+p\frac{\partial z}{\partial y}=0.$$

$$z \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) + p \frac{\partial z}{\partial x} = 0$$

$$z \frac{\partial^2 z}{\partial y \partial x} + p \frac{\partial z}{\partial x} = 0$$

$$zs + pq = 0$$

Which is the required partial diff. equation of second order.

From the partial differential equation of $ax+by+cz=0$.

Given that $ax+by+cz=0$ —— (1)

Where a, b, c are arbitrary constants.

Diff (1) partially w.r.t x and y, we get

$$a + cp = 0 \implies a = -cp \text{ —— (2)}$$

$$b + cq = 0 \implies b = -cq \text{ —— (3)}$$

Sub. (2) and (3) in (1), we get

$$-cpx + cqy + cz = 0$$

$$z = px + qy$$

Which is the required partial differential equation.

Form a partial differential equation by eliminating the arbitrary constants a, b, c from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— ①

Diff ① w.r.t 'x', partially, we get.

$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}=0 \Rightarrow \frac{c^2}{a^2}x=-z\frac{\partial z}{\partial x} \quad —— ②$$

Diff ② w.r.t 'x', partially, we get.

$$\frac{c^2}{a^2}=-\left[z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\right] \quad —— ③$$

From ②, $\frac{c^2}{a^2}=-\frac{z}{x}\frac{\partial z}{\partial x}$ —— ④.

From ③ and ④, we get.

$$-\frac{z}{x}\frac{\partial z}{\partial x}=-\left[z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\right]$$

$$\frac{z}{x}\frac{\partial z}{\partial x}=z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2. \quad —— ⑤$$

This is required partial differential equation.

Note:- Diff. the given equation partially w.r.t 'y' twice and eliminating $\frac{c^2}{b^2}$, we obtain the equation. $\frac{z}{y}\frac{\partial z}{\partial y}=z\frac{\partial^2 z}{\partial y^2}+\left(\frac{\partial z}{\partial y}\right)^2$ —— ⑥ as another required partial differential equation.

Remark:—

Partial differential equations obtained by elimination of arbitrary constants are not unique.

→ Find a partial differential equation representing all planes that are at a constant perpendicular distance p from the origin.

Sol.- The equation of a plane that is at a perpendicular distance p from the origin is given by. $lx+my+nz=p$ —— ①.

Where (l,m,n) are the direction cosines of the normal to the plane which satisfy the identity $l^2+m^2+n^2=1$ —— ②.

As (l,m,n) take different values, subject to the identity ②, equation ① represents different planes all of whose perpendicular distance from the origin is p. In otherwords. ① is the cartesian equation of the given set (family) of planes.

Diff ① w.r.t x and y, we get.

$$l + n\frac{\partial z}{\partial x} = 0, \quad m + n\frac{\partial z}{\partial y} = 0.$$

$$l = -n\frac{\partial z}{\partial x} \qquad m = -n\frac{\partial z}{\partial y} \quad \text{---} \quad ③$$

Sub. these in ②, we get

$$n^2\left(\frac{\partial z}{\partial x}\right)^2 + n^2\left(\frac{\partial z}{\partial y}\right)^2 + n^2 = 1$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{1}{n^2} \quad \text{---} \quad ④$$

Now substituting for l and m from ③ in ①, we get

$$\left(-n\frac{\partial z}{\partial x}\right)x + \left(-n\frac{\partial z}{\partial y}\right)y + nz = p.$$

$$z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{p}{n}.$$

$$\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)^2 = \frac{p^2}{n^2}$$

Sub. for $\frac{1}{n^2}$ in ④, we get.

$$\left(z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)^2 = p^2\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

This is the partial differential equation that represents the given set of planes.

Note :- (i) In case of arbitrary functions the order of partial diff. equation is equal to the number of arbitrary functions involved and to be eliminated from the given equation.

(ii) If the given relation b/n x, y, z contains two arbitrary functions then leaving a few exceptional cases the partial differential eqn's of higher order than the second will be formed.

(1) Form the partial differential equation of the following by eliminating arbitrary functions (a) $z = f(x^2-y^2)$ (b) $z = f(x+at) + g(x-at)$

Sol:- (a) Given that $z = f(x^2-y^2)$ —— (1)

Where f is an arbitrary function.

Here the given equation has one arbitrary function so we obtain a first order partial differential equation.

Diff (1) partially w.r.t x, we have

$$p = \frac{\partial z}{\partial x} = f'(x^2-y^2) \cdot 2x.$$

$$p = f'(x^2-y^2) \cdot 2x \quad \text{—— (2)}$$

Diff (1) partially w.r.t y, we have.

$$q = \frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y)$$

$$q = f'(x^2-y^2)(-2y) \quad \text{—— (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'(x^2-y^2) \cdot 2x}{f'(x^2-y^2)(-2y)}$$

$$py + qx = 0.$$

Which is the required first order partial differential equation.

(b) Given that $z = f(x+at) + g(x-at)$ —— (1)

Where f, g are arbitrary functions

Here the equation contains two arbitrary functions so we obtain P.D.E of second order.

Diff (1) partially w.r.t 'x', we get

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at) \quad \text{—— (2)}$$

Again diff partially w.r.t 'x', we have.

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \quad \text{—— (3)}$$

Diff (1) partially w.r.t 't', we have.

$$\frac{\partial z}{\partial t} = f'(x+at) \cdot a + g'(x-at)(-a)$$

Again diff. partially w.r.t t, we have

$$\frac{\partial^2 z}{\partial t^2} = f''(x+at)\, a^2 + g''(x-at)\, a^2$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 [f''(x+at) + g''(x-at)]$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \quad [\because \text{from (3)}]$$

Which is the required partial diff. eqn of 2nd order.

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Note :- If the implicit form of the equation is $f(u,v)=0$ then its explicit form can be written as $u=f(v)$ or $v=f(u)$.

) Form the differential equation of the following by eliminating arbitrary functions (a) $f(x^2+y^2, z-xy)=0$ (b) $xyz=\phi(x+y+z)$

Sol:- (a) Given that $f(x^2+y^2, z-xy)$

which is the implicit form of the equation

Its explicit form is $x^2+y^2=f(z-xy)$ — ①

where f is an arbitrary function and treat z as a function of x and y

Diff ① partially w.r.t 'x', we have

$$2x = f'(z-xy)\left(\frac{\partial z}{\partial x}-y\right)$$

$$2x = f'(z-xy)(p-y) \quad \text{— ②}$$

Diff ① partially w.r.t 'y', we have.

$$2y = f'(z-xy)\left(\frac{\partial z}{\partial y}-x\right)$$

$$2y = f'(z-xy)(q-x) \quad \text{— ③}$$

$$\frac{②}{③} \Rightarrow \frac{2x}{2y} = \frac{f'(z-xy)(p-y)}{f'(z-xy)(q-x)}$$

$$(q-x)x = (p-y)y$$

$$qx-py = x^2-y^2$$

Which is the 1st order P.D.E.

1b) Sol:- Given that $xyz = \phi(x+y+z)$ —— ①

Which is in explicit form

Where ϕ is an arbitrary function and treat z is a function of x.

Diff ① partially w.r.t x

$$y\left[x\frac{\partial z}{\partial x} + z\right] = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right).$$

$$y(px+z) = \phi'(x+y+z)(1+p) \text{ —— ②}.$$

Diff ① partially w.r.t y, we have

$$x\left[y.\frac{\partial z}{\partial y} + z\right] = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

$$x(yq+z) = \phi'(x+y+z)(1+q) \text{ —— ③}$$

$$\frac{②}{③} \Rightarrow \frac{y(px+z)}{x(z+yq)} = \frac{\phi'(x+y+z)(1+p)}{\phi'(x+y+z)(1+q)}$$

$$y(px+z)(1+q) = x(z+yq)(1+p)$$

$$x(y-z)p + y(z-x)q = z(x-y)$$

Which is the required partial diff. eqn.

12) Form a partial differential equation by eliminating the arbitrary functions $f(x)$ and $g(y)$ from $z = yf(x) + x\,g(y)$

Sol:- Given that $z = yf(x) + x\,g(y)$ —— ①.

Diff ① w.r.t x and y partially, we have

$$\frac{\partial z}{\partial x} = yf'(x) + g(y)$$

$$p = yf'(x) + g(y) \text{ —— ②}$$

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$$\frac{\partial z}{\partial y} = f(x) + x g'(y)$$

$$q = f(x) + x g'(y) \text{ --- } (3)$$

Since the relations (1), (2) and (3) are not sufficient to eliminate f, g, f' and g' so we find the second order partial derivatives.

$$\frac{\partial^2 z}{\partial x^2} = r = y f''(x) \text{ --- } (4)$$

$$\frac{\partial^2 z}{\partial x \partial y} = s = f'(x) + g'(y) \text{ --- } (5).$$

$$\frac{\partial^2 z}{\partial y^2} = t = x g''(y) \text{ --- } (6)$$

From (2) and (3), we have

$$\left.\begin{aligned} f'(x) &= \frac{1}{y}\left[p - g(y)\right] \\ g'(y) &= \frac{1}{x}\left[q - f(x)\right] \end{aligned}\right\} \text{ --- } (7)$$

sub. (7) in (5), we get

$$s = \frac{1}{y}\left[p - g(y)\right] + \frac{1}{x}\left[q - f(x)\right]$$

$$xys = x\left[p - g(y)\right] + y\left[q - f(x)\right]$$

$$xys = px + qy - \left[x g(y) + y f(x)\right]$$

$$xys = px + qy - z$$

which is the required partial diff equation.

Linear Partial Differential Equations of the first order :-

A differential equation containing first order partial derivatives p and q only is called a partial differential equation of the 1^{st} order [OR] If p and q occur only in the first degree and are not multiplied together, then it is called a linear partial differential equation of 1^{st} order.

Eg:- (i) $2p + 30q = 44z$ is a linear P.D.E.

(ii) $2p^2 + 3044q^2 = 1$ is a Non linear.

Lagrange's Linear Equation :-

A linear partial differential equation of order one, involving a dependent variable z and two independent variables x and y, of the form

$$Pp + Qq = R.$$

Where P, Q, R are functions of x, y, z is called Lagranges linear equation.

General Solution of the Linear Equation :-

We have seen that from a relation $\phi(u,v) = 0$ — ①, a linear partial differential equation $Pp + Qq = R$ — ② is derived by eliminating the arbitrary function ϕ. Suppose that ② has been derived from ① then $\phi(u,v) = 0$ is called the general solution or a general integral of the equation ②.

Working Procedure to solve $Pp + Qq = R$:-

Step 1 :- Compare the given partial differential equation with $Pp + Qq = R$ and identity P, Q and R.

Step 2 :- Write the subsidiary equations or auxiliary equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Step 3 :- Find any two independent solutions of the subsidiary equations Let the two solutions be $u = a$ and $v = b$. Where a and b are constants.

Step.4: Now the general solution of $Pp + Qq = R$ is given by $f(u,v) = 0$ or $u = f(v)$ or $v = f(u)$.

Method of grouping :-

From the Lagrange's subsidiary equations, if the variables are separated by taking any two members.

i.e $\frac{dx}{P} = \frac{dy}{Q}$ [OR] $\frac{dy}{Q} = \frac{dz}{R}$ [OR] $\frac{dz}{R} = \frac{dx}{P}$.

Then by integrating, the solution $u(x,y) = c_1$ or $v(y,z) = c_2$ or $w(z,x) = c_3$ is obtained. Then by taking any two of these solutions we can write the complete solution of Lagrange's equation.

i.e $\phi(u,v) = 0$ or $\phi(v,w) = 0$ or $\phi(u,w) = 0$

1) Solve $px + qy = z$

Sol:- Given that $px + qy = z$ —— (1)

Compare equation (1) with $Pp + Qq = R$

$$P = x, \quad Q = y, \quad R = z$$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Considering the first two members.

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating bothsides, we get

$$\int \frac{dx}{x} = \int \frac{dy}{y} + \log c$$

$$\log|x| = \log|y| + \log c$$

$$\log\left|\frac{x}{y}\right| = \log c$$

$$\frac{x}{y} = c = u(x,y)$$

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Considering the last two members,

we have $\frac{dy}{y} = \frac{dz}{z}$

Integrating both sides, we get

$$\int \frac{dy}{y} = \int \frac{dz}{z} + \log c_1$$

$$\log|y| = \log|z| + \log c_1$$

$$\log\left|\frac{y}{z}\right| = \log c_1$$

$$\frac{y}{z} = c_1 = v(y,z)$$

$\therefore$ The complete solution of given eqn is $\phi(u,v)=0$

i.e $\phi\left(\frac{x}{y}, \frac{y}{z}\right)=0$.

(2) Solve $p\tan x + q\tan y = \tan z$.

Sol: Given that $p\tan x + q\tan y = \tan z$

The auxiliary equations are $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

Taking first two members, we have

$$\cot x\, dx = \cot y\, dy$$

Integrating bothsides, we get

$$\int \cot x\, dx = \int \cot y\, dy + \log c_1$$

$$\log|\sin x| = \log|\sin y| + \log c_1$$

$$\log\left|\frac{\sin x}{\sin y}\right| = \log c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

Taking last two members, we have

$$\cot y\, dy = \cot z\, dz$$

Integrating bothsides we get

$$\int \cot y\, dy = \int \cot z\, dz + \log c_2$$

$$\log|\sin y| = \log|\sin z| + \log c_2$$

$$\log\left|\frac{\sin y}{\sin z}\right| = \log c_2$$

$$\frac{\sin y}{\sin z} = c_2$$

$\therefore$ The general solution of ① is

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0 .$$

Method of Multipliers :—

Consider the Lagrange's auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} .$$

By an algebraic principle choosing a proper set of multipliers l, m, n which are not necessarily constants.

We have each ratio $= \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ such that $lP + mQ + nR = 0$.

$\Longrightarrow l\,dx + m\,dy + n\,dz = 0$ ($\because$ on cross multiplication)

on integrating we obtain the first solution of Lagrange's linear equation $u(x, y, z) = a$ —— ①.

Similarly choosing l', m', n' as another set of multipliers (which are not necessarily constants) we have

$$\text{Each ration} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R} .$$

Such that $l'P + m'Q + n'R = 0$.

$$\Longrightarrow l'dx + m'dy + n'dz = 0 .$$

On integrating, we obtain the another solution of Lagrange's equation i.e $v(x,y,z) = c_2$ — (2).

Hence the complete solution of Lagranges equation is

$$\phi(u,v) = 0 \quad \text{or} \quad u = f(v) \quad \text{or} \quad v = f(u).$$

Note:- (i) Both the methods i.e Grouping and multipliers can be applied to solve the problem.

(ii) The selection of multipliers depends upon the functions P, Q, R involved in the equation. The multipliers l, m, n are called Lagrangian multipliers.

(1) Solve $x(y-z)p + y(z-x)q = z(x-y)$

Sol:- Given that $x(y-z)p + y(z-x)q = z(x-y)$.

This is a Lagranges linear equation $Pp + Qq = R$

$$P = x(y-z) \quad Q = y(z-x) \quad R = z(x-y).$$

The Lagrange's auxiliary equations are.

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

By taking 1, 1, 1 as first set of multipliers, we get

$$\text{Each ratio} = \frac{1dx + 1dy + 1dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$= \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0 \quad (\because \text{on cross multiplication})$$

Integrating

$$\int (dx + dy + dz) = c_1$$

$$x + y + z = c_1 = u(x,y,z).$$

Again by taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as another set of multipliers,

we have Each ratio $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y-z) + \frac{1}{y}y(z-x) + \frac{1}{z}z(x-y)}$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$ ($\because$ on cross multiplication)

Integrating

$$\int \left(\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz\right) = \log c_2$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2 = v(x,y,z)$$

The complete solution of given equation is $\phi(u,v) = 0$

i.e $\phi(x+y+z, xyz) = 0$.

(2) Find the partial diff. eqn $(y+z)p + (z+x)q = x+y$.

Sol:- Given that $(y+z)p + (z+x)q = x+y$.

Here $P = y+z$ $Q = z+x$ $R = x+y$.

The Lagranges subsidary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}.$$

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(i) Taking 1, -1, 0 as multipliers, we get

Each fraction is also $= \dfrac{dx-dy}{y-x}$. — (2)

(ii) Taking 0, 1, -1 as multipliers, we get

Each fraction is also $= \dfrac{dy-dz}{z-y}$. — (3)

(iii) Taking 1, 1, 1 as multipliers, we get

Each fraction is also $= \dfrac{dx+dy+dz}{2(x+y+z)}$ — (4)

From (1), (2) (3) and (4), we get

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}.$$

Taking 4th and 5th members.

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

Integrating both sides, we get

$$\int \frac{dx-dy}{x-y} = \int \frac{dy-dz}{y-z} + \log c_1.$$

$$\log|x-y| = \log|y-z| + \log c_1.$$

$$\log\left|\frac{x-y}{y-z}\right| = \log c_1$$

$$(x-y) = c_1(y-z)$$

Taking 5th and 6th members

$$\frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

Integrating bothsides, we get

$$-2\int \frac{dy-dz}{y-z} = \int \frac{dx+dy+dz}{x+y+z} + \log c_2$$

$$-2\log|y-z| = \log|x+y+z| + \log c_2$$

$$\log\left|\frac{1}{(y-z)^2}\right| = \log|c_2(x+y+z)|$$

$$\frac{1}{c_2} = (x+y+z)(y-z)^2 .$$

$\therefore$ The solution of the given partial differential equation

$$\phi(c_1, c_2) = 0 \text{ i.e. } \phi\left(\frac{x-y}{y-z}, (y-z)^2(x+y+z)\right) = 0 .$$

Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$.

Sol:- Given that $(x^2-yz)p+(y^2-zx)q=z^2-xy$ —— ①

Compare eqn ① with $Pp+Qq=R$.

$P=x^2-yz \qquad Q=y^2-zx \qquad R=z^2-xy$.

The Lagrange's Auxiliary equations are.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

i.e $$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}.$$

(i) Taking 1, -1, 0 as one set of multipliers.

$$\text{Each ratio} = \frac{dx-dy}{(x^2-yz)-(y^2-zx)}.$$

$$=\frac{dx-dy}{(x^2-y^2)+z(x-y)}=\frac{dx-dy}{(x-y)(x+y+z)} \quad \text{—— ②}$$

(ii) Taking 0, 1, -1 as another set of multipliers

$$\text{Each ratio} = \frac{dy-dz}{(y^2-zx)-(z^2-xy)}$$

$$=\frac{dy-dz}{(y^2-z^2)+x(y-z)}=\frac{dy-dz}{(y-z)(x+y+z)} \quad \text{—— ③}$$

From ② and ③, we get

$$\text{Each ratio} = \frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$

$$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

Integrating bothsides, we get

$$\int \frac{dx - dy}{x - y} = \int \frac{dy - dz}{y - z} + \log c_1$$

$$\log|x-y| - \log|y-z| = \log c_1$$

$$\log\left|\frac{x-y}{y-z}\right| = \log c_1$$

$$\frac{x-y}{y-z} = c_1$$

(iii) Taking x, y, z as a set of multipliers, we get

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2-yz) + y(y^2-zx) + z(z^2-xy)} = \frac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{---} \ (4)$$

(iv) Taking $1, 1, 1$ as another set of multipliers, we get

$$\text{Each ratio} = \frac{dx + dy + dz}{(x^2-yz) + (y^2-zx) + (z^2-xy)} = \frac{dx + dy + dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{---} \ (5)$$

From (4) and (5), we get

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx + dy + dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx + dy + dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)(dx + dy + dz)$$

Integrating bothsides, we get

$$\int x\,dx + \int y\,dy + \int z\,dz = \int (x+y+z)(dx + dy + dz) + c_2$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} - \frac{(x+y+z)^2}{2} = c_2$$

$$\frac{(x^2+y^2+z^2) - (x+y+z)^2}{2} = c_2$$

$$xy + yz + zx = -c_2$$

$$xy + yz + zx = c_3 \quad \text{where } c_3 = -c_2 .$$

$\therefore$ The complete sol. of given P.D.E is.

$$\phi\left(\frac{x-y}{y-z},\; xy + yz + zx\right)$$

Solve. $p\cos(x+y) + q\sin(x+y) = z$.

sol:- Given that $p\cos(x+y) + q\sin(x+y) = z$ —— ①

Compare equation ① with $Pp + Qq = R$.

Here $P = \cos(x+y)$ $\quad Q = \sin(x+y)$ $\quad R = z$.

The Lagrange's Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

i.e $$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \text{ —— ②}$$

(i) Taking 1, 1, 0 as a set of multipliers, we get.

Each ratio $= \dfrac{dx+dy}{\cos(x+y)+\sin(x+y)}$ —— ③

(ii) Taking 1, −1, 0 as another set of multipliers, we get.

Each ratio $= \dfrac{dx-dy}{\cos(x+y)-\sin(x+y)}$ —— ④

From ②, ③ and ④, we get.

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} = \frac{dx+dy}{\sin(x+y)+\cos(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

→ Taking 3rd and 4th members, we get.

$$\frac{dz}{z} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)}$$

put $x+y = t \Rightarrow dx + dy = dt$.

$$\frac{dz}{z} = \frac{dt}{\cos t + \sin t}$$

$$\frac{dz}{z} = \frac{dt}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right)}$$

$$\sqrt{2}\,\frac{dz}{z} = \frac{dt}{\sin\frac{\pi}{4}\cos t + \cos\frac{\pi}{4}\sin t}$$

$$\sqrt{2}\,\frac{dz}{z} = \frac{dt}{\sin\left(t+\frac{\pi}{4}\right)}$$

$$\sqrt{2}\,\frac{dz}{z} = \operatorname{cosec}\left(t+\frac{\pi}{4}\right)dt.$$

Integrating bothsides, we get.

$$\sqrt{2}\int\frac{dz}{z} = \int \operatorname{cosec}\left(t+\frac{\pi}{4}\right)dt + \log c_1$$

$$\sqrt{2}\log|z| = \log\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1.$$

$$\log|z|^{\sqrt{2}} = \log c_1 \tan\frac{1}{2}\left(t+\frac{\pi}{4}\right)$$

$$z^{\sqrt{2}} = c_1 \tan\frac{1}{2}\left(t+\frac{\pi}{4}\right)$$

$$z^{\sqrt{2}}\cot\frac{1}{2}\left(x+y+\frac{\pi}{4}\right) = c_1 \quad \left[\because x+y=t\right]$$

→ Taking 4^{th} and 5^{th} members, we get

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

$$\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx+dy) = dx-dy.$$

[put $x+y=t$

$dx+dy=dt$]

$$\frac{\cos t-\sin t}{\cos t+\sin t}\,dt = dx-dy.$$

Integrating bothsides, we get

$$\int\frac{\cos t-\sin t}{\cos t+\sin t}\,dt = \int dx - \int dy + \log c_2$$

$$\log|\cos t+\sin t| - \log c_2 = x-y$$

$$\log\left|\frac{\cos t+\sin t}{c_2}\right| = x-y.$$

$$\frac{\cos t+\sin t}{c_2} = e^{x-y}$$

$$e^{-(x-y)}(\cos t+\sin t) = c_2$$

$\therefore$ The complete sol. of given P.D.E is

$$\phi\left(z^{\sqrt{2}}\,\cancel{\cot}\,\tfrac{1}{2}\left(x+y+\tfrac{\pi}{4}\right),\ e^{y-x}(\cos t+\sin t)\right) = 0.$$

Solve $(x^3+3xy^2)p + (y^3+3x^2y)q = 2(x^2+y^2)z$.

sol: Given that $(x^3+3xy^2)p + (y^3+3x^2y)q = 2(x^2+y^2)z$ —— (1)

Compare eqn (1) with $Pp+Qq=R$.

Here $P = x^3+3xy^2$ $\quad Q = y^3+3x^2y$ $\quad R = 2(x^2+y^2)z$

The Lagrange's Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)} \quad \text{—— (2)}$$

(i) choose 1, 1, 0 as one set of multipliers, we get

$$\text{Each ratio} = \frac{dx+dy}{x^3+3xy^2+y^3+3x^2y}$$

$$= \frac{dx+dy}{(x+y)^3} \quad \text{—— (3)}$$

(ii) choose 1, −1, 0 as another set of multipliers, we get

$$\text{Each ratio} = \frac{dx-dy}{x^3+3xy^2-y^3-3x^2y}$$

$$= \frac{dx-dy}{(x-y)^3} \quad \text{—— (4)}$$

(iii) choose $\frac{1}{x}, \frac{1}{y}, 0$ as 3^{rd} set of multipliers, we get

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy}{\frac{1}{x}(x^3+3xy^2)+\frac{1}{y}(y^3+3x^2y)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy}{4(x^2+y^2)} \quad \text{--- (5)}$$

From (2), (3), (4) and (5), we get

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)} = \frac{dx+dy}{(x+y)^3} = \frac{dx-dy}{(x-y)^3} = \frac{\frac{1}{x}dx+\frac{1}{y}dy}{4(x^2+y^2)}$$

→ Taking ~~5th~~ and 4^{th} members, we get.

$$\frac{dx+dy}{(x+y)^3} = \frac{dx-dy}{(x-y)^3}.$$

Integrating bothsides, we get

$$\int\frac{dx+dy}{(x+y)^3} - \int\frac{dx-dy}{(x-y)^3} = c_1.$$

$$-\frac{1}{2(x+y)^2} + \frac{1}{2(x-y)^2} = c_1.$$

→ Taking 3^{rd} and 6^{th} members, we get

$$\frac{dz}{2z(\cancel{x^2+y^2})} = \frac{\frac{1}{x}dx + \frac{1}{y}dy}{4(\cancel{x^2+y^2})}$$

Integrating bothsides, we get

$$\int\frac{dz}{z} - \frac{1}{2}\left[\int\frac{1}{x}dx + \int\frac{1}{y}dy\right] = \log c_2$$

$$\log|z| - \frac{1}{2}\left[\log|x| + \log|y|\right] = \log c_2$$

$$\log|z| - \frac{1}{2}\log|xy| = \log c_2$$

$$\log\left|\frac{z}{\sqrt{xy}}\right| = \log c_2$$

$$\frac{z}{\sqrt{xy}} = c_2$$

The complete sol. of given P.D.E is $\phi\left(\frac{z}{\sqrt{xy}}, \frac{1}{2(x-y)^2} - \frac{1}{2(x+y)^2}\right) = 0$.

Non Linear Partial Differential Equations of First Order :—

A Partial differential equation in which the partial derivatives p and q occur other than the first degree and are multiplied together is said to be a non linear partial differential eqn. of first order.

Complete Integral or Complete solution :—

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete solution of the given equation.

Particular Integral :—

A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral.

Singular Integral :— Let $f(x,y,z,p,q)=0$ be a partial differential equation whose complete integral is $\phi(x,y,z,a,b)=0$ —— ①.

Differentiating ① partially w.r.t a and b and then equate to zero, we get $\frac{\partial \phi}{\partial a}=0$ —— ② and $\frac{\partial \phi}{\partial b}=0$. —— ③.

Eliminating a and b by using equations ①, ② and ③.

The eliminant of a and b is called singular integral.

Solutions of a partial differential equation :-

Consider partial differential equations of first order involving two inde-pendent variables x and y and the dependent variable z. Such an equation is of the form. $f(x, y, z, p, q) = 0$ —— ①.

In this equation, p and q may appear with first or higher powers or in product forms. If p and q appear only with first powers and there occur no products among themselves, the equation is called a linear equation; otherwise it is called a non linear equation.

Complete Solution :-

Suppose $\phi(x, y, z, a, b) = 0$ —— ② is a relation from which the partial diffe-rential equation ① is derived by eliminating the arbitrary constants a and b. Then ② is called a complete solution (integral) of equation ①.

For given values of a and b, relation ② represents a surface. As a and b are chosen arbitrary, this relation represents a two parameter family of surfaces.

Eg:- Eliminating the arbitrary constants a and b, the relation

$z = (x^2+a)(y^2+b)$ —— ① yields a partial differential equation

$pq = 4xyz$ —— ②

∴ The relation ① is a complete solution of the 1^{st} order P.D.E

Particular Solution :-

Suppose we give particular values to the arbitrary constants a and b present in the complete solution $\phi(x, y, z, a, b) = 0$ of equation $f(x, y, z, p, q) = 0$. Then $\phi(x, y, z, a, b) = 0$ becomes a particular solution (integral) of equation $f(x, y, z, p, q) = 0$. This solution represents a particular member of the family of surfaces given by the complete solution $\phi(x, y, z, a, b) = 0$.

Eg:- $z = (x^2+a)(y^2+b)$ is a complete solution of $pq = 4xyz$.
①

if we put $a=2, b=1$ in solution ①, we get $z = (x^2+2)(y^2+1)$. as a particular solution of equation $pq = 4xyz$. This represents a particular surface of the family of surfaces given by the complete solution $z = (x^2+a)(y^2+b)$.

General Solution :-

Suppose in the complete solution $\phi(x,y,z,a,b)=0$ ① we take b as known function of a say $b=\psi(a)$ (or vice versa) Then ① becomes.

$$\phi(x,y,z,a,\psi(a)) = 0 \quad \text{——} ②$$

Diff ② partially w.r.to 'a', we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial \psi}\psi'(a) = 0 \quad \text{——} ③$$

Suppose we eliminate "a" from relations ② and ③, if the elimination is possible. The resulting relation, if it satisfies $f(x,y,z,p,q)=0$, is called a general solution (integral) of equation $f(x,y,z,p,q)=0$.

This solution represents the envelope of the one parameter family of surfaces represented by $\phi(x,y,z,a,\psi(a))=0$.

For example :- $z = (x^2+a)(y^2+b)$ ① is complete solution of $pq = 4xyz$ ②.

Suppose we take $b=a$ Then we get

$$z = (x^2+a)(y^2+a) \quad \text{——} ③$$

$$x^2y^2 + a(x^2+y^2) + a^2 - z = 0 \quad \text{——} ④$$

Differentiating this partially, w.r.t 'a', we get

$$x^2+y^2+2a = 0$$

$$\Rightarrow a = -\frac{1}{2}(x^2+y^2) \quad \text{——} ⑤$$

Sub ⑤ in ④, we get

$$x^2y^2 - \frac{1}{2}(x^2+y^2)^2 + \frac{1}{4}(x^2+y^2)^2 - z = 0.$$

$$(x^2-y^2)^2 + 4z = 0.$$

This is a general solution of equation ②.

Singular Solution :-

Let $\phi(x,y,z,a,b)=0$ — (1) be a complete solution of $f(x,y,z,p,q)=0$. — (2)

Diff (1) w.r. to. "a" and 'b'. partially, we get.

$$\frac{\partial\phi}{\partial a}=0, \quad \frac{\partial\phi}{\partial b}=0 \quad \text{— (3)}$$

Suppose it is possible to eliminate a and b from relations (1) and (3). Then the resulting relation, if it satisfies equation (2). is called the singular solution (integral) of equation (2). This solution represents the envelope of the two - parameter family of surfaces represented by the complete solution. $\phi(x,y,z,a,b)=0$.

Eg:- $z=(x^2+a)(y^2+b)$ — (1) is complete solution of $pq=4xyz$ — (2)

Diff (1) w.r.t 'a' and 'b' partially, we get

$$y^2+b=0, \quad x^2+a=0.$$

Substituting for a and b from these in (1), we get

$$z=0 \quad \text{— (3)}$$

This satisfies equation (2).

Hence this is the singular solution of equation (2).

Special types of Non linear equations of first order :-

Standard form 1 :-

An equation is of the form $f(p,q)=0$ i.e Equations involving only p and q and no x, y, z.

Let the required solution be

$$z = ax + by + c \quad \text{--- (1)}$$

Diff (1) w.r.t x and y partially, we get

$$\frac{\partial z}{\partial x} = a \quad \text{and} \quad \frac{\partial z}{\partial y} = b$$

$p = a$ and $q = b$

Sub. these values in $f(p,q)=0$, we get

$f(a,b)=0$.

From this we can obtain b in terms of a.

Let $b = \phi(a)$.

Then (1) becomes $z = ax + \phi(a)y + c$ --- (2).

This relation contains two arbitrary constants a and c and is therefore a complete solution of equation (1).

If c is a specified function of a, say $c = \psi(a)$, then (2) becomes

$$z = ax + \phi(a)y + \psi(a) \quad \text{--- (3)}$$

Diff (3) w.r.t 'a' partially, we get

$$0 = x + \phi'(a)y + \psi'(a). \quad \text{--- (4)}$$

Eliminating a from (3) and (4), we get a general solution of equation (1).

Diff (2) w.r.t a and c partially, we get

$$0 = a + \psi'(a)y \quad \text{--- (5)}$$

$$0 = 1. \quad \text{--- (6)}$$

To get the singular solution, we have to eliminate a and c from relations (2), (5) and (6).

Since (6) is absurd, we infer that there is no singular solution for equation (1).

Solve the equation $q = 3p^2$. Also, find its general solution that passes through (contains) the point $(-1,0,0)$.

Sol:- Given that $q = 3p^2$ —— ①

This is of the form $f(p,q) = 0$.

The complete solution of given P.D.E is $z = ax + by + c$ —— ②

Diff ② w.r.t x and partially we get

$$\frac{\partial z}{\partial x} = a \quad \text{i.e } p = a.$$

$$\frac{\partial z}{\partial y} = b \quad \text{i.e } q = b.$$

Sub. p and q in ①, we get $b = 3a^2$.

$$z = ax + 3a^2y + c.$$

If this solution contains the point $(-1,0,0)$, it should be satisfied for $x = -1, y = 0, z = 0$. that is we should have $c = a$, and the solution becomes $z = a(x+1) + 3a^2y$ —— ③.

Diff ③ w.r.t 'a' partially, we get.

$$0 = (x+1) + 6ay. \implies a = -\frac{(x+1)}{6y}.$$

Sub. a in ③, we get

$$z = -\frac{(x+1)}{6y}(x+1) + 3\left(-\frac{(x+1)}{6y}\right)^2 y.$$

$$z = -\frac{(x+1)^2}{12y}.$$

$$(x+1)^2 + 12yz = 0.$$

This is the general solution of the given equation, that passes through the point $(-1,0,0)$.

Solve $p^2+q^2=npq$

Sol:- Given that $p^2+q^2=npq$ —— ①

Let the required solution be $z=ax+by+c$ —— ②

Diff ② w.r.t 'x' and 'y' partially, we get

$$\frac{\partial z}{\partial x}=a \Rightarrow p=a$$

$$\frac{\partial z}{\partial y}=b \Rightarrow q=b$$

Sub. these values in ①, we get.

$$a^2+b^2-nab=0 \text{ —— ③}$$

To find complete integral, we have to eliminate any one of the arbitrary constants

From ③. We have. $b^2-nab+a^2=0$.

Which is a quadratic equation in b.

$$\therefore\ b=\frac{na\pm\sqrt{n^2a^2-4a^2}}{2}$$

$$=\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right] \text{ —— ④}$$

Sub. ④ in ①, we get.

$$z=ax+\frac{ay}{2}\left[n\pm\sqrt{n^2-4}\right]+c.$$

This is the complete integral because it contains only two arbitrary constants.

Standard form 2 :-

Equation of the form $f(z,p,q)=0$ i.e Equations containing z, p, q only and no x, y :

Let us assume that z is a function of u and $u = x+ay$.

$$z = f(u) \quad \text{and} \quad u = x+ay .$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

[$\because$ z is a function of single independent variable u, so we use ordinary derivative $\frac{dz}{du}$ and u is a function of x, y. so we use partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$].

Substituting these values of p and q in the given equation, we get $f\left(z, \frac{dz}{du}, a\frac{dz}{\partial u}\right)=0$

Which is an ordinary differential equation of the first order and it can be solved.

Finally, replace $u = x+ay$ to get the complete solution of the given equation.

Working Procedure :—

Step 1 :- Assume $z = f(u)$ and $u = x+ay$ so that

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du} .$$

Step 2 :- Substitute the values of p and q in the given equation

Step 3 :- Solve the resulting ordinary differential equation in z and u

Step 4 :- Replace $u = x + ay$ in the complete solution.

Note :- To solve an equation of the above form, we also assume that $z = f(u)$ and $u = y + ax$ Then $p = \frac{\partial z}{\partial x} = a\frac{dz}{du}$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}.$$

(1) Solve $z^2 = 1 + p^2 + q^2$.

Sol:- Given that the equation of the form $f(z, p, q) = 0$.

Let $z = f(u)$ where $u = x + ay$ be the solution of the given equation.

Then $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

Substituting these values of p and q in the given equation, we have.

$$z^2 = 1 + \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{dy}\right)^2$$

$$z^2 - 1 = \left(\frac{dz}{du}\right)^2 (1 + a^2)$$

$$\sqrt{z^2 - 1} = \frac{dz}{du}\sqrt{1 + a^2}$$

Separating the variables and integrating both sides, we get

$$\int \frac{dz}{\sqrt{z^2 - 1}} = \frac{1}{\sqrt{1 + a^2}}\int du + c.$$

$$\cosh^{-1} z = \frac{u}{\sqrt{1 + a^2}} + c.$$

Hence the solution is.

$$\cosh^{-1}(z) = \frac{x + ay}{\sqrt{1 + a^2}} + c.$$

Verify that $z = ae^{b(x+by)}$ is a complete solution of the equation $p^2 = qz$. Show that $z=0$ is both a particular solution and the singular solution.

Sol:- Given that $z = ae^{b(x+by)}$ ——— (1)

Diff (1) w.r.t 'x' and 'y' partially, we get.

$$p = \frac{\partial z}{\partial x} = ab\, e^{b(x+by)} = bz \text{ ——— (2)}$$

$$q = \frac{\partial z}{\partial y} = ab^2 e^{b(x+by)} = b^2 z \text{ ——— (3)}$$

From (2) and (3)

$$qz = b^2 z^2 = (bz)^2$$

$$qz = p^2$$

$$\therefore \; p^2 = qz \text{ ——— (4)}$$

$p^2 = qz$ is a P.D.E obtained by eliminating the constants a and b from the relation. In other words (1) is a complete solution of equation (4).

For $a=0$, the complete solution (1) becomes $z=0$.

Thus, $z=0$ is a particular solution.

Diff (1) w.r.t 'a' and 'b' partially, we get

$$0 = e^{b(x+by)} \text{ ——— (5)}$$

$$0 = a\, e^{b(x+by)} (x+2by) \text{ ——— (6)}$$

Relations (1), (5) and (6) yield $z=0$. This obeys equation (4).

Thus $z=0$ is the singular solution also.

Standard form 3 :-

Equation of the form $f_1(x,p) = f_2(y,q)$.

i.e Equation in which z is absent and the term involving x and p can be separated from those involving y and q.

Let $f_1(x,p) = f_2(y,q) = a$ [constant]

Now solve these equations for p and q by taking $f_1(x,p) = a$ and $f_2(y,q) = a$.

Solving for p and q, we obtain.

$$p = F_1(x,a) \text{ and } q = F_2(y,a)$$

$$\text{since } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = p\,dx + q\,dy$$

sub. the values of p and q in above equation.

$$dz = F_1(x,a)\,dx + F_2(y,a)\,dy.$$

Integrating both sides, we get

$$\int dz = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + c$$

$$z = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + c.$$

Which is the required complete solution of the given equation.

1 Solve $pq + qx = y$

Sol:- Given that $pq + qx + y$.

This is of the form $f(p,q,x,y) = 0$.

The given equation can be written as.

$$q(p+x) = y \quad \text{i.e } p+x = \frac{y}{q}.$$

Let $p+x = \frac{y}{q} = a$ (constant) Then.

$$p+x = a \quad \text{and } \frac{y}{q} = a$$

$$p = a-x \qquad q = \frac{y}{a}.$$

We know that $dz = p\,dx + q\,dy$

$$dz = (a-x)\,dx + \frac{y}{a}\,dy.$$

Integrating bothsides, we get

$$\int dz = \int (a-x)\,dx + \frac{1}{a}\int y\,dy + c.$$

$$z = ax - \frac{x^2}{2} + \frac{1}{a}\frac{y^2}{2} + c.$$

$$2az = a^2x - ax^2 + y^2 + 2ac.$$

$$2az = a^2x - ax^2 + y^2 + c_1 \quad \text{where } c_1 = 2ac.$$

Which is the complete integral.

Solve the equation $z^2(p^2z^2+q^2)=1$. Show that this equation has no singular solution.

Sol:- Given that $z^2(p^2z^2+q^2)=1$ ——— (1).

This is of the form $f(p,q,z)=0$.

Let $z=f(u)$, $u=x+ay$ be the complete solution of (1)

$u=x+ay$

$\frac{\partial u}{\partial x}=1$

$\frac{\partial u}{\partial y}=a$.

$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}\frac{\partial u}{\partial x}=\frac{dz}{du}.$$

$$q=\frac{\partial z}{\partial y}=\frac{dz}{du}\frac{\partial u}{\partial y}=a\frac{dz}{du}$$

Sub. p and q in (1), we get

$$z^2\left(\frac{dz}{du}\right)^2(z^2+a^2)=1$$

$$\frac{dz}{du}=\pm\frac{1}{z\sqrt{z^2+a^2}}$$

$$\pm z\sqrt{z^2+a^2}\,dz=du.$$

Integrating bothsides, we get

$$\pm\int z\sqrt{z^2+a^2}\,dz=\int du+b.$$

$$\pm\frac{(z^2+a^2)^{3/2}}{3}=u+b.$$

$$(z^2+a^2)^{3/2}=3(x+ay+b)$$

$$(z^2+a^2)^3=9(x+ay+b)^2 \text{ ——— } (*)$$

This is a complete solution of the given P.D.E ——— (2).

Diff (2) w.r.t a and b partially, we get

$$18(x+ay+b)y=6(z^2+a^2)^2a \text{ ——— } (3)$$

$$18(x+ay+b)=0 \text{ ——— } (4).$$

From these, we get $a=0$ and $x+ay+b=0$.

Putting these in (*), we get $z=0$.

We check that $z=0$ does not satisfy the given equation.

$\therefore$ $z=0$ is not the singular solution.

Thus, the given equation has no singular solution.

Solve the equation $q^2 = p^2 z^2 (1-p^2)$. Show that $z=0$ is the singular solution.

Sol:- Given that $f(z) = q^2 = p^2 z^2 (1-p^2)$ —— ①.

This is of the form $f(p,q,z)=0$.

Let $z = f(u)$ where $u = x+ay$ be the complete sol. of ①

$u = x+ay$

$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = a.$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}.$$

Sub. p and q in ①, we get.

$$\left(a \frac{dz}{du}\right)^2 = \left(\frac{dz}{du}\right)^2 z^2 \left(1-\left(\frac{dz}{du}\right)^2\right)$$

$$a^2 = z^2 \left(1-\left(\frac{dz}{du}\right)^2\right)$$

$$\frac{dz}{du} = \pm \frac{\sqrt{z^2-a^2}}{z}$$

$$\pm \frac{z\,dz}{\sqrt{z^2-a^2}} = du.$$

Integrating bothsides, we get

$$\pm \frac{1}{2}\int \frac{2z\,dz}{\sqrt{z^2-a^2}} = \int du + b.$$

$$\pm \sqrt{z^2-a^2} = u+b.$$

$$z^2-a^2 = (x+ay+b)^2. \quad \text{—— ②}.$$

This is the complete solution of the given equation.

Diff ② w.r.t a and b partially, we get

$$-2a = 2(x+ay+b)y, \quad 0 = 2(x+ay+b)$$

These relations give $x+ay+b=0$ and $a=0$.

Putting these in ①, we get $z=0$.

Which satisfies the given relation.

$\therefore$ $z=0$ is the singular solution of the given equation.

Standard form IV Clairaut's Form :—

Equations of the form $z = px + qy + f(p,q)$. Is called Clairaut's equation

The complete solution of the equation $z = px + qy + f(p,q)$ is

$z = ax + by + f(a,b)$.

Let the required solution be $z = ax + by + c$ Then.

$p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$ and putting for a, b in ①.

Note:- For the solution of $z = px + qy + f(p,q)$ replace p by a and q by b.

(1) Prove that the complete integral of $z = px + qy + \sqrt{p^2+q^2+1}$ represents all planes at unit distance from the origin.

Sol:- The given equation is of the form $z = px + qy + f(p,q)$.

A Complete solution is $z = ax + by + \sqrt{a^2+b^2+1}$.

or $ax + by - z + \sqrt{a^2+b^2+1} = 0$. —①.

Which is a family of planes.

The length of the perpendicular drawn from the origin to the planes ①

$$= \frac{\sqrt{a^2+b^2+1}}{\sqrt{a^2+b^2+1}} = 1.$$

Verify that $z = ax^2 + by^2 + ab$ is a complete solution of the equation $pq + 2xy(px+qy) = 4xyz$ and show that $z = 1 + 2xy$ is the general solution containing the point $(0,0,1)$.

Sol. Given that $z = ax^2 + by^2 + ab$ ——①.

Diff ① w.r.t x and y partially, we get

$$p = \frac{\partial z}{\partial x} = 2ax, \quad q = \frac{\partial z}{\partial y} = 2by$$

Sub. for a and b from these in ①, we get

$$z = \left(\frac{p}{2x}\right)x^2 + \left(\frac{q}{2y}\right)y^2 + \frac{pq}{4xy}$$

$$pq + 2xy(px+qy) = 4xyz \quad \text{--- (2)}$$

This is a P.D.E obtained by eliminating the arbitrary constants a and b from the relation (1).

In other words, (1) is a complete solution of equation (2).

If the solution (1) contains the point $(0,0,1)$, we get $ab = 1$.

Which gives $b = \frac{1}{a}$.

$$z = ax^2 + \frac{y^2}{a} + 1 \quad \text{--- (3)}$$

Diff. this partially w.r.t 'a', we obtain

$$0 = x^2 - \frac{y^2}{a^2}$$

so that $a = \frac{y}{x}$

Putting this in (3), we get $z = 1 + 2xy$.

This is the general solution of (2) containing the point $(0,0,1)$.

Standard form IV Clairaut's Form :-

Equations of the form $z = px + qy + f(p,q)$ is called clairauts eqn. —— (1)

The complete sol. of the equation $z = px + qy + f(p,q)$ is

$z = ax + by + f(a,b)$. —— (2)

Let the required solution be $z = ax + by + c$ Then.

$p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$ and putting for a, b in (1).

<u>Note</u> :- For the solution of $z = px + qy + f(p,q)$ replace p by a and q by b.

→ When b is a specified fun. of a say $b = \phi(a)$, the solution (2) becomes $z = ax + \phi(a)y + f(a, \phi(a))$. —— (3)

Diff. (3) w.r.t 'a', partially, we get.

$$0 = x + \phi'(a)y + f'(a) \quad \text{—— (4)}$$

Eliminating a between (3) and (4). we get a general sol. of the given equation.

→ Diff (2) w.r.to a and b partially, we get.

$$0 = x + \frac{\partial f}{\partial a}, \quad 0 = y + \frac{\partial f}{\partial y} \quad \text{—— (5)}$$

Eliminating a and b from these relations and (2), we get the singular sol. of the given equation.

(1) Prove that the complete integral of $z = px + qy + \sqrt{p^2+q^2+1}$ represents all planes at unit distance from the origin.

Sol: Given that $z = px + qy + \sqrt{p^2+q^2+1}$

The given equation is of the form $z = px + qy + f(p,q)$.

Which is the clairaut's equation.

A complete sol. of given equation is obtained by replacing p by a and q by b, we get.

$$z = ax + by + \sqrt{a^2+b^2+1}.$$

$$\text{(or)} \quad ax + by - z + \sqrt{a^2+b^2+1} = 0. \quad \text{——} (1)$$

Which represents a family of planes:

The length of the ⊥er drawn from the origin to the planes (1) is.

$$= \frac{\sqrt{a^2+b^2+1}}{\sqrt{a^2+b^2+1}} = 1.$$

Find the complete solution of the equation $pqz = p^2(qx+p^2) + q^2(py+q^2)$

Sol:- Given that $pqz = p^2(qx+p^2) + q^2(py+q^2)$

$$pqz = p^2qx + p^4 + q^2py + q^4.$$

$$z = p\frac{x}{\cancel{p}} + \frac{p^3}{q} + qy + \frac{q^3}{p}.$$

$$z = px + qy + \left(\frac{p^3}{q} + \frac{q^3}{p}\right)$$

Which is of the form $z = px + qy + f(p,q)$.

Which is the clairaut's equation..

A complete sol. of given equation is obtained by replacing p by a q by b, we get.

$$z = ax + by + \left(\frac{a^3}{b} + \frac{b^3}{a}\right).$$

Solve $z = px + qy + p^2 - q^2$.

sol:- Given that $z = px + qy + p^2 - q^2$.

Which is in the clairaut's form.

The complete solution is $z = ax + by + a^2 - b^2$ —— (1).

General Integral:

Put $b = \phi(a)$ in (1), we get

$z = ax + \phi(a)y + a^2 - [\phi(a)]^2$ —— (2).

Diff (2) w.r.t 'a', we get

$0 = x + \phi'(a).y + 2a - 2\phi(a)\phi'(a)$ —— (3).

Eliminating a between (2) and (3), we get the general integral.

Singular Integral:

Diff (1) w.r.t a and b.

$0 = x + 2a. \Rightarrow a = \frac{-x}{2}$.

$0 = y - 2b \Rightarrow b = \frac{y}{2}$.

Sub. in (1), $z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2 - y^2}{4}$.

$4z = y^2 - x^2$. is the singular solution.

Solve $z = px + qy + p^2 q^2$.

sol:- Given that $z = px + qx + p^2 q^2$.

Which is clairaut's form.

The complete solution is $z = ax + by + a^2 b^2$ —— (1).

Singular Integral:-

Diff (1) w.r.t a and b.

$0 = x + 2ab^2 \Rightarrow x = -2ab^2$

$0 = y + 2b^2 a. \Rightarrow y = -2b^2 a$.

$\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{c}$ say.

$a = cy, \; b = cx$.

Sub. in ②, we get

$$x = -2c^3yx^2$$

$$c^3 = \frac{-1}{2xy}$$

Sub. a and b in ①.

$$z = cxy + cxy + c^4x^2y^2.$$

$$z = 2cxy + cx^2y^2\left(\frac{-1}{2xy}\right)$$

$$z = 2cxy - \frac{c}{2}xy = \frac{3}{2}cxy$$

$$z^3 = \frac{27}{8}c^3x^3y^3.$$

$$z^3 = \frac{27}{8}x^3y^3\left(\frac{-1}{2xy}\right)$$

$$z^3 = -\frac{27}{16}x^2y^2.$$

$16z^3 + 27x^2y^2 = 0$ is the singular integral.

General Integral :-

Put $b = \phi(a)$ in ①.

$$z = ax + \phi(a)y + a^2[\phi(a)]^2 \text{ --- } ③.$$

Diff ③ w.r.t a and eliminating a, the general integral can be obtained.

Solve $z = px + qy + \log pq$.

Sol: Given that $z = px + qy + \log pq$.

This is clairaut's form.

The complete sol. is $z = ax + by + \log ab$. --- ①

Singular integral Diff ① w.r.t a and b partially, we get

$$0 = x + \frac{1}{a} \Rightarrow a = -\frac{1}{x}, \quad 0 = y + \frac{1}{b} \Rightarrow b = -\frac{1}{y}.$$

Sub. in ① the singular integral is $z = -1 - 1 + \log\left(\frac{1}{xy}\right)$

$$z = -2 - \log xy.$$

General Integral :- Put $b = \phi(a)$ in ①.

$$z = ax + \phi(a)y + \log a\,\phi(a) \text{ --- } ②$$

Diff ② w.r.t a and eliminating a the general integral can be obtained.

Solve $z = px + qy - 2\sqrt{pq}$.

Sol: Given that $z = px + qy - 2\sqrt{pq}$.

This is in clairaut's form

The complete Integral is $z = ax + by - 2\sqrt{ab}$ —— (1).

Singular Integral:-

Diff (1) w.r.t a and b partially, we get.

$$0 = x - \frac{2}{2\sqrt{ab}}\, b \quad \text{i.e} \quad x = \sqrt{\frac{b}{a}}$$

$$0 = y - \frac{2}{2\sqrt{ab}}\, a \quad \text{i.e} \quad y = \sqrt{\frac{a}{b}}$$

Eliminating a and b, the singular integral is $xy = 1$.

General Integral:-

Let $b = \phi(a)$ where ϕ is an arbitrary function.

Sub. in (1), we get $z = ax + \phi(a)y - 2\sqrt{a\phi(a)}$ —— (2).

Diff (2) w.r.t 'a', we get.

$$0 = x + \phi'(a)y - 2\left[\sqrt{a}\,\frac{1}{2}[\phi(a)]^{-\frac{1}{2}}\phi'(a) + [\phi(a)]^{\frac{1}{2}}\frac{1}{2}a^{-\frac{1}{2}}\right]$$

$$0 = x + y\phi'(a) - \left[\sqrt{a}\,\frac{\phi'(a)}{\sqrt{\phi(a)}} + \frac{\sqrt{\phi(a)}}{\sqrt{a}}\right]$$

The general integral can be obtained by eliminating 'a' from (2) and (3).

Solve $z = px + qy + c\sqrt{1+p^2+q^2}$.

Sol: Given that $z = px + qy + c\sqrt{1+p^2+q^2}$.

Which is in clairaut's form.

The Complete Integral is $z = ax + by + c\sqrt{1+a^2+b^2}$ —— (1).

Singular Integral:-

Diff (1) w.r.t a and b. partially, we get.

$$0 = x + \frac{ca}{\sqrt{1+a^2+b^2}} \quad \text{—— (2)}$$

$$0 = y + \frac{cb}{\sqrt{1+a^2+b^2}} \quad \text{—— (3)}$$

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$$x^2+y^2 = \frac{c^2(a^2+b^2)}{1+a^2+b^2}$$

$$c^2-(x^2+y^2) = c^2 - \frac{c^2(a^2+b^2)}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

$$1+a^2+b^2 = \frac{c^2}{c^2-(x^2+y^2)}$$

$$\frac{1}{\sqrt{1+a^2+b^2}} = \frac{\sqrt{c^2-(x^2+y^2)}}{c}$$

$$x = \frac{-ac}{\sqrt{1+a^2+b^2}} = \frac{-ac\sqrt{c^2-x^2-y^2}}{c}$$

$$x = -a\sqrt{c^2-x^2-y^2}$$

$$a = \frac{-x}{\sqrt{c^2-x^2-y^2}}$$

Similarly $b = \dfrac{-y}{\sqrt{c^2-x^2-y^2}}$

Substituting these values of a and b in ①, we get

$$z = \frac{-x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$z = \sqrt{c^2-x^2-y^2}$$

$$x^2+y^2+z^2 = c^2$$

This is the singular Integral.

General Integral:

Put $b=\phi(a)$ in ①.

$$z = ax+\phi(a)y + c\sqrt{1+a^2+[\phi(a)]^2} \quad \text{—— ④}$$

Diff ④ w.r.t a and eliminating a, general integral can be obtained.

Verify that $z = ax^2 + by^2 + ab$ is a complete sol. of the equation $pq + 2xy(px+qy) = 4xyz$ and show that $z = 1+2xy$ is the general solution containing the point $(0,0,1)$.

Sol: Given that $z = ax^2 + by^2 + ab$ —— ①

Diff ① w.r.t x and y partially, we get.

$$p = \frac{\partial z}{\partial x} = 2ax \quad , \quad q = \frac{\partial z}{\partial y} = 2by.$$

$$a = \frac{p}{2x} \qquad b = \frac{q}{2y}.$$

Sub. a and b ~~from~~ in eqn ①, we get

$$z = \left(\frac{p}{2x}\right)x^2 + \left(\frac{q}{2y}\right)y^2 + \frac{pq}{4xy}$$

$$z = \frac{px}{2} + \frac{qy}{2} + \frac{pq}{4xy}$$

$$4xyz = 2px^2y + 2qxy^2$$

$$4xyz = 2xy(px+qy). \text{ —— ②}$$

This is a P.D.E obtained by eliminating the arbitrary constants a and b from the relation ①.

In otherwords, ① is a complete sol. of eqn ②.

If the sol. ① contains the point $(0,0,1)$, we get $ab = 1$.

which gives $b = \frac{1}{a}$.

$$z = ax^2 + \frac{y^2}{a} + 1 \text{ —— ③}$$

Diff ③ w.r.t 'a' partially, we get

$$0 = x^2 - \frac{y^2}{a^2}$$

so that $a = \frac{y}{x}$ —— ④

Sub. ④ in ③, we get $z = 1+2xy$.

This is the general sol. of ② containing the point $(0,0,1)$.

Equations Reducible to standard forms :—

Type 1 :- Equations of the type $f(x^m p, y^n q)=0$ where m and n are constants : — ①

The above form of the equation can be transformed to an equation of the form $f(P,Q)=0$ by the substitution.

Case (i) :- When $m \neq 1$ and $n \neq 1$

Put $X = x^{1-m}$ and $Y = y^{1-n}$ then.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}$$

$$X = x^{1-m}$$
$$\frac{\partial X}{\partial x} = (1-m)x^{-m}.$$

$$p = P(1-m)x^{-m}$$

$$x^m p = (1-m)P$$

Where $P = \frac{\partial z}{\partial X}$.

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$

$$Y = y^{1-n}$$
$$\frac{\partial Y}{\partial y} = (1-n)y^{-n}$$

$$q = Q(1-n)y^{-n}$$

Where $Q = \frac{\partial z}{\partial Y}$.

$$y^n q = Q(1-n)$$

Now the given equation ① reduces to $f((1-m)P, (1-n)Q)=0$.

Which is of the form $f(P,Q)=0$. (standard form – 1) — ②

Let the complete sol. of P.D.E ② be of the form

$$z = aX + bY + c \quad \text{—— ③}$$

Diff. ③ w.r.t X and Y partially, we get

$$P = \frac{\partial z}{\partial X} = a, \quad Q = \frac{\partial z}{\partial Y} = b.$$

Sub. P and Q in ②, we get $f(a,b)=0$ — ④

Using ④, Write b interms of a i.e $b=\phi(a)$ ⑤

Sub. ⑤ in ③, we get $z = aX + \phi(a)Y + c.$ ⑥

Finally replace X with x^{1-m} and Y with y^{1-n} in ⑥, we get the required complete solution of ①.

Case(ii) When $m=1$ and $n=1$

The given P.D.E is of the form $f(x^m p, y^n q)=0$.

i.e $f(xp, yq)=0$ —— ①.

Put $X=\log x$ and $Y=\log y$ then.

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}$$

$X=\log x$

$\frac{\partial X}{\partial x}=\frac{1}{x}$.

$$p=P\cdot\frac{1}{x}$$

Where $\frac{\partial z}{\partial X}=P$.

$$xp=P.$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}.$$

$Y=\log y$

$\frac{\partial Y}{\partial y}=\frac{1}{y}$.

$$q=Q.\frac{1}{y}$$

Where $\frac{\partial z}{\partial Y}=Q$.

$$yq=Q.$$

Now the given equation ① reduces to $f(P, Q)=0$. —— ②.
(standard form – 1)

Let the complete sol. of P.D.E ② of the form $z=aX+bY+c$ —— ③

Diff ③ w.r.t X and Y partially, we get.

$$P=\frac{\partial z}{\partial X}=a \qquad Q=\frac{\partial z}{\partial Y}=b.$$

Sub. P and Q in ③, we get $f(a,b)=0$ —— ④

Using ④, Write b interms of a i.e $b=\phi(a)$ —— ⑤.

Sub. ⑤ in ③, we get $z=aX+\phi(a)Y+c$ —— ⑥.

Finally replace X with $\log x$ and Y with $\log y$ in ⑥, we get the required complete solution of ①.

Equations Reducible to standard forms:—

(i) Equations of the type $f(x^m p, y^n q) = 0$ where m and n are constants.

The above form of the equation can be transformed to an equation of the form $f(P, Q) = 0$ by the substitution.

Case (i):- When $m \neq 1$ and $n \neq 1$

Put $X = x^{1-m}$ and $Y = y^{1-n}$ then.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P(1-m)x^{-m} \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$x^m p = P(1-m).$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q(1-n)y^{-n} \quad \text{where } Q = \frac{\partial z}{\partial Y}$$

$$y^n q = Q(1-n)$$

Now the given equation reduces to $f[(1-m)P, (1-n)Q] = 0$.

Which is of the form $f(P, Q) = 0$.

Case (ii):- When $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$ Then.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x}.$$

$$px = P \quad \text{where } P = \frac{\partial z}{\partial X}.$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{1}{y}$$

$$qy = Q \quad \text{where } Q = \frac{\partial z}{\partial Y}.$$

Now the given equation reduces to the form $f(P, Q) = 0$

(1) Solve $px + qy = 1$.

Sol:- Given that $px + qy = 1$ —— (1).

This is of the form $f(x^m p, y^n q) = 0$.

Here $m = 1$, $n = 1$.

Put $\log x = X$ and $\log y = Y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot \frac{1}{x}$$

$$P = px. \quad \text{—— (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q \cdot \frac{1}{y}$$

$$Q = qy \quad \text{—— (3)}$$

Sub. (2) and (3) in (1), we get

$$P + Q = 1 \quad \text{—— (4)}$$

Eqn (4) is of the form $f(P, Q) = 0$.

The complete sol. of $f(P, Q) = 0$ is $z = aX + bY + c$ —— (5)

Diff (5) w.r.t X and Y partially, we get

$$\frac{\partial z}{\partial X} = a \quad \text{i.e} \quad P = a.$$

$$\frac{\partial z}{\partial Y} = b \quad \text{i.e} \quad Q = b$$

Sub. the values of P and Q in (4), we get

$$a + b = 1.$$

$$b = 1 - a.$$

$\therefore$ The general solution of (1) is

$$z = aX + (1-a)Y + c.$$

$$z = a\log x + (1-a)\log y + c.$$

Solve $\frac{x^2}{p} + \frac{y^2}{q} = z$.

Sol:- Given that $\frac{x^2}{p} + \frac{y^2}{q} = z$

$$x^2 p^{-1} + y^2 q^{-1} = z$$

$$(x^{-2} p)^{-1} + (y^{-2} q)^{-1} = 1 \quad \text{—— ①}$$

This is of the form $f(x^m p, y^n q) = 0$.

Here $m = -2$, $n = -2$.

Put $X = x^{1-m}$ $\qquad Y = y^{1-n}$

$X = x^{1-(-2)} = x^3$ $\qquad Y = y^{1-(-2)} = y^3$

$\frac{\partial X}{\partial x} = 3x^2$ $\qquad \frac{\partial Y}{\partial y} = 3y^2$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot 3x^2 \implies x^{-2} p = 3P \quad \text{—— ②}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q \cdot 3y^2 \implies y^{-2} q = 3Q \quad \text{—— ③}$$

Sub. ② and ③ in ①, we get.

$$(3P)^{-1} + (3Q)^{-1} = 1$$

$$\frac{1}{P} + \frac{1}{Q} = 3 \quad \text{—— ④}$$

This is of the form $f(P, Q) = 0$ (standard form - I)

Let the complete solution of ④ is $z = aX + bY + c$.

$\frac{\partial z}{\partial X} = a$ i.e $P = a$ —— ⑤

$\frac{\partial z}{\partial Y} = b$ i.e $Q = b$ —— ⑥

Sub. ⑤ & ⑥ in ④, we get.

$$\frac{1}{a} + \frac{1}{b} = 3 \implies \frac{1}{b} = \left(3 - \frac{1}{a}\right)$$

$$\frac{1}{b} = \frac{3a-1}{a} \implies b = \frac{a}{3a-1}.$$

$$\therefore \quad z = ax + \frac{a}{3a-1} y + c.$$

$\therefore$ The Complete solution of given P.D.E is

$$z = ax^3 + \frac{a}{3a-1} \cdot y^3 + c.$$ Where a, c are arbitrary constants.

Equations of the type $f(x^m p, y^n q, z) = 0$ where m and n are Constants

This can be reduced to an equation of the form $f(P, Q, z) = 0$ by the substitutions given for the equation $f(x^m p, y^n q) = 0$.

(1) Solve $x^2p^2 + y^2q^2 = z^2$.

Sol:- Given that $x^2p^2 + y^2q^2 = z^2$.

It can be written as $(xp)^2 + (yq)^2 = z^2$ —— ①.

This is of the form $f(z, x^m p, y^n q) = 0$.

Here $m=1$, $n=1$.

Put $X = \log x$ and $Y = \log y$ Then,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot \frac{1}{x} \quad \text{where } P = \frac{\partial z}{\partial X}$$

$$P = px \text{ —— ②}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y} = Q \cdot \frac{1}{y} \quad \text{where } Q = \frac{\partial z}{\partial Y}$$

$$Q = qy. \text{ —— ③}$$

Sub. ② and ③ in ①, we get

$$P^2 + Q^2 = z^2 \text{ —— ④}$$

Eqn ④ is of the form $f(z, P, Q) = 0$.

Let $z = f(u)$ where $u = X + aY$ be the solution ④

Then $P = \frac{dz}{du}$ and $Q = a\frac{dz}{du}$.

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2.$$

$$\left(\frac{dz}{du}\right)^2 (1+a^2) = z^2.$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}.$$

Separate the variables and integrate.

$$\int \frac{dz}{z} = \frac{1}{\sqrt{1+a^2}} \int du + c.$$

$$\log z = \frac{u}{\sqrt{1+a^2}} + c.$$

$\therefore$ The complete Integral is

$$\log z = \frac{X+aY}{\sqrt{1+a^2}} + c.$$

$$\log z = \frac{\log x + a\log y}{\sqrt{1+a^2}} + c.$$

(2) Solve $2x^4p^2 - yzq - 3z^2 = 0$.

Sol:- Given that $2x^4p^2 - yzq - 3z^2 = 0$

It can be written as $2(x^2p)^2 - yzq - 3z^2 = 0$ —— (1)

This is of the form $f(z, x^m p, y^n q) = 0$.

Here $m=2$, $n=1$.

Put $X = x^{1-2} = x^{-1}$ and $Y = \log y$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\left(-\frac{1}{x^2}\right)$$

$$-P = px^2. \quad \text{—— (2)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$$

$$Q = qy \quad \text{—— (3)}$$

Sub (2) and (3) in (1), we get

$$2P^2 - Qz - 3z^2 = 0 \quad \text{—— (4)}$$

This is of the form $f(z, P, Q) = 0$ —— (5).

Let $z = f(u)$ where $u = X + aY$ be the sol. of (4).

Then $P = \frac{dz}{du}$ and $Q = a\frac{dz}{du}$ —— (5).

sub ⑤ in ④, we get.

$$2\left(\frac{dz}{du}\right)^2 - a\frac{dz}{du}z - 3z^2 = 0.$$

$$2\left(\frac{dz}{du}\right)^2 - az\frac{dz}{dy} - 3z^2 = 0$$

This is a quadratic equation in $\frac{dz}{du}$.

$$\frac{dz}{du} = \frac{az \pm \sqrt{a^2z^2 + 24z^2}}{4}$$

$$\frac{dz}{du} = \frac{z\left[a \pm \sqrt{a^2+24}\right]}{4}$$

Separate the variables and Integrate

$$\int\frac{dz}{z} = \frac{a \pm \sqrt{a^2+24}}{4}\int du + c.$$

$$\log z = \frac{a \pm \sqrt{a^2+24}}{4}u + c.$$

The general Sol. is.

$$\log z = \frac{a \pm \sqrt{a^2+24}}{4}(x+ay) + c.$$

$$\log z = \frac{a \pm \sqrt{a^2+24}}{4}\left(\frac{1}{x} + a\log y\right) + c.$$

(2) solve $z^2(p^2+q^2) = x^2+y^2$.

Sol:- Given that $z^2(p^2+q^2) = x^2+y^2$.

The given diff. eqn. can be written as $(pz)^2 - x^2 = y^2 - (qz)^2$ — ①.

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=1$.

Put $Z = z^{n+1}$

$$Z = z^{1+1} = z^2.$$

$$Z = z^2.$$

$$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x}$$

$$P = 2zp$$

$$zp = \frac{P}{2}.$$

$$\frac{\partial Z}{\partial y} = 2z\frac{\partial z}{\partial y}$$

$$Q = 2zq$$

$$zq = \frac{Q}{2}.$$

sub. the values of zp and zq in ①, weget.

$$\left(\frac{P}{2}\right)^2 - x^2 = y^2 - \left(\frac{Q}{2}\right)^2$$

$$\frac{P^2}{4} - x^2 = y^2 - \frac{Q^2}{4}.$$

This is of the form $f_1(x, P) = f_2(y, Q)$.

$$\frac{P^2}{4} - x^2 = y^2 - \frac{Q^2}{4} = a.$$

$$\frac{P^2}{4} - x^2 = a \qquad y^2 - \frac{Q^2}{4} = a.$$

$$P^2 = 4a + 4x^2 \qquad Q^2 = 4y^2 - 4a.$$

$$P = 2\sqrt{x^2+a}. \qquad Q = 2\sqrt{y^2-a}$$

We have $dz = P\,dx + Q\,dy$.

$$dz = 2\sqrt{x^2+a}\,dx + 2\sqrt{y^2-a}\,dy.$$

Integrating bothsides, we get

$$\int dz = 2\int\sqrt{x^2+a}\,dx + 2\int\sqrt{y^2-a}\,dy + c.$$

$$z = 2\left[\frac{x}{2}\sqrt{x^2+a} + \frac{a}{2}\sinh^{-1}\left(\frac{x}{\sqrt{a}}\right)\right] + 2\left[\frac{y}{2}\sqrt{y^2-a} + \frac{a}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right] + c.$$

$$z = x\sqrt{x^2+a} + a\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2-a} - a\cosh^{-1}\left(\frac{y}{a}\right) + c.$$

$\therefore$ The complete solution of (1) is

$$z^2 = x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\sinh^{-1}\left(\frac{x}{a}\right) - a\cosh^{-1}\left(\frac{y}{a}\right) + c.$$

Equations of the type $f_1(x, pz^n) = f_2(y, qz^n)$ where n is constant

An equation of the above form can be reduced to an equation of the form $f_1(x, P) = f_2(y, Q)$ by the substitutions

$$\text{Put} \quad Z = \begin{cases} z^{n+1} & \text{if } n \neq -1 \\ \log z & \text{if } n = -1 \end{cases}$$

1) solve $(x+pz)^2 + (y+qz)^2 = 1$.

Sol:- Given that $(x+pz)^2 + (y+qz)^2 = 1$.

The given diff. eqn can be written as

$$(x+pz)^2 = 1-(y+qz)^2$$

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=1$

Put $Z = z^{n+1}$

$$Z = z^{1+1} = z^2$$

$$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x}$$

$$\frac{\partial Z}{\partial x} = 2zp$$

$$zp = \frac{1}{2}\frac{\partial Z}{\partial x}$$

$$zp = \frac{1}{2}P.$$

$$\frac{\partial Z}{\partial y} = 2z\frac{\partial z}{\partial y}.$$

$$\frac{\partial Z}{\partial y} = 2zq$$

$$zq = \frac{1}{2}Q.$$

Sub. the values of pz and qz given equations. becomes...

$$\left(x+\frac{P}{2}\right)^2+\left(y+\frac{Q}{2}\right)^2=1.$$

$$\left(x+\frac{P}{2}\right)^2=1-\left(y+\frac{Q}{2}\right)^2 \quad \text{———} \quad (2)$$

This is of the form $f_1(x,P)=f_2(y,Q)$.

$$\left(x+\frac{P}{2}\right)^2=1-\left(y+\frac{Q}{2}\right)^2=a^2.$$

$$\left(x+\frac{P}{2}\right)^2=a \qquad 1-\left(y-\frac{Q}{2}\right)^2=a.$$

$$x+\frac{P}{2}=\sqrt{a} \qquad y-\frac{Q}{2}=\sqrt{1-a}.$$

$$x=\sqrt{a}+\frac{P}{2} \qquad Q=2\left[y-\sqrt{1-a}\right]$$

$$P=2\left(x-\sqrt{a}\right)$$

We have $dz=P\,dx+Q\,dy$

$$dz=2(x-\sqrt{a})\,dx+2\left[y-\sqrt{1-a}\right]dy$$

Integrating bothsides, weget

$$\int dz=2\int(x-\sqrt{a})\,dx+2\int\left[y-\sqrt{1-a}\right]dy+c$$

$$z=x^2-2\sqrt{a}\,x+y^2-\sqrt{1-a}.\,y+c.$$

$\therefore$ The Complete solution of (1) is

$$z^2=x^2-2\sqrt{a}\,x+y^2-\sqrt{1-a}\,y+c.$$

Solve $p^2z^2\sin^2x + q^2z^2\cos^2y = 1$.

sol:- Given that $p^2z^2\sin^2x + q^2z^2\cos^2y = 1$. —— ①

It can be written as $(pz)^2\sin^2x + (q^1z)^2\cos^2y = 1$.

$$(pz)^2\sin^2x = 1-(qz)^2\cos^2y \quad \text{—— ②}$$

This is of the form. $f(x, pz^n) = g(y, qz^n)$ with $n=1$.

Put $Z = z^{n+1}$.

$Z = z^2$

$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x}$ | $\frac{\partial Z}{\partial y} = 2z\frac{\partial z}{\partial y}$.

$P = 2zp$ | $Q = 2z.q$

$zp = \frac{P}{2}$ | $zq = \frac{Q}{2}$.

Sub. the values of pz and qz in ②, we get

$$\left(\frac{P}{2}\right)^2\sin^2x = 1-\left(\frac{Q}{2}\right)^2\cos^2y.$$

$$P^2\sin^2x = 4-Q^2\cos^2y.$$

This is of the form $f_1(x,P) = f_2(y,Q)$.

Let $P^2\sin^2x = 4-Q^2\cos y = a^2$.

Then $P^2\sin^2x = a^2$ and $4-Q^2\cos^2y = a^2$.

$P^2 = a^2\csc^2x$ and $Q^2\cos^2y = 4-a^2$

$P = a\csc x$. and $Q^2 = (4-a^2)\cos^2y$

$P = a\csc x$ and $Q = \sqrt{4-a^2}\cos y$.

We know that $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$

$$dZ = P\,dx + Q\,dy$$

$$dZ = a\csc x\,dx + \sqrt{4-a^2}\cos y\,dy$$

Solve $p^2z^2\sin^2x + q^2z^2\cos^2y = 1$.

sol:- Given that $p^2z^2\sin^2x + q^2z^2\cos^2y = 1$. —— ①

It can be written as $(pz)^2\sin^2x + (q z)^2\cos^2y = 1$

$$(pz)^2\sin^2x = 1-(qz)^2\cos^2y \quad \text{—— ②}$$

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=1$.

Put $Z = z^{n+1}$

$Z = z^2$

$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x}$ | $\frac{\partial Z}{\partial y} = 2z\frac{\partial z}{\partial y}$

$P = 2zp$ | $Q = 2z.q$

$zp = \frac{P}{2}$ | $zq = \frac{Q}{2}$.

Sub. the values of pz and qz in ②, we get

$$\left(\frac{P}{2}\right)^2\sin^2x = 1-\left(\frac{Q}{2}\right)^2\cos^2y$$

$$P^2\sin^2x = 4-Q^2\cos^2y.$$

This is of the form $f_1(x, P) = f_2(y, Q)$.

Let $P^2\sin^2x = 4-Q^2\cos y = a^2$.

Then $P^2\sin^2x = a^2$ and $4-Q^2\cos^2y = a^2$.

$P^2 = a^2\csc^2x$ and $Q^2\cos^2y = 4-a^2$

$P = a\csc x$. and $Q^2 = (4-a^2)\cos^2y$

$P = a\csc x$ and $Q = \sqrt{4-a^2}\cos y$.

We know that $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$

$$dZ = P\,dx + Q\,dy$$

$$dZ = a\csc x\,dx + \sqrt{4-a^2}\cos y\,dy$$

Integrating bothsides, we get.

$$\int dz = a\int \csc x\, dx + \sqrt{1-a^2}\int \sec y\, dy + c.$$

$$Z = a\log|\csc x - \cot x| + \sqrt{1-a^2}\log|\sec y + \tan y| + c.$$

$$z^2 = a\log|\csc x - \cot x| + \sqrt{1-a^2}\log|\sec y + \tan y| + c$$

Which is the complete sol. of the given P.D.E $\left[\because Z = z^2\right]$

Solve $z(p^2-q^2) = x-y$.

sol:- Given that $z(p^2-q^2) = x-y$. — (1)

It can be written as $(\sqrt{z}\,p)^2 - (\sqrt{z}\,q)^2 = x-y$.

$$(z^{1/2}p)^2 - x = (z^{1/2}q)^2 - y \quad \text{— (2)}.$$

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n = \frac{1}{2}$.

Put $Z = z^{n+1}$

$$Z = z^{\frac{1}{2}+1}$$

$$Z = z^{3/2}.$$

$$\frac{\partial Z}{\partial x} = \frac{3}{2} z^{1/2} \cdot \frac{\partial z}{\partial x}.$$

$$P = \frac{3}{2} z^{1/2} p.$$

$$z^{1/2}p = \frac{2}{3}P \quad \text{— (3)}$$

$$\frac{\partial Z}{\partial y} = \frac{3}{2} z^{1/2} \frac{\partial z}{\partial y}.$$

$$Q = \frac{3}{2} z^{1/2} q$$

$$z^{1/2}q = \frac{2}{3}Q. \quad \text{— (4)}$$

Sub. (3) and (4) in (2), we get

$$\left(\frac{2}{3}P\right)^2 - x = \left(\frac{2}{3}Q\right)^2 - y.$$

$$\frac{4}{9}P^2 - x = \frac{4}{9}Q^2 - y.$$

This is of the form $f_1(x, P) = f_2(y, Q)$.

Let $\frac{4}{9}P^2 - x = \frac{4}{9}Q^2 - y = a$.

Then $\frac{4}{9}P^2 - x = a$ and $\frac{4}{9}Q^2 - y = a$.

$\frac{4}{9}P^2 = a + x$ and $\frac{4}{9}Q^2 = a + y$

$P^2 = \frac{9}{4}(a+x)$ and $Q^2 = \frac{9}{4}(a+y)$

$P = \frac{3}{2}(a+x)^{1/2}$ and $Q = \frac{3}{2}(a+y)^{1/2}$.

We know that $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$.

$dZ = P\,dx + Q\,dy$.

$dZ = \frac{3}{2}(a+x)^{1/2}dx + \frac{3}{2}(a+y)^{1/2}dy$

Integrating bothsides, we get

$\int dZ = \frac{3}{2}\int (a+x)^{1/2}dx + \frac{3}{2}\int (a+y)^{1/2}dy + c.$

$Z = \frac{3}{2}\cdot\frac{2}{3}(a+x)^{3/2} + \frac{3}{2}\cdot\frac{2}{3}(a+y)^{3/2} + c.$

$Z = (a+x)^{3/2} + (a+y)^{3/2} + c.$

$z^{3/2} = (a+x)^{3/2} + (a+y)^{3/2} + c \quad [\because Z = z^{3/2}]$

Which is the complete sol. of given P.D.E.

Form the partial differential equation by eliminating arbitrary functions from $f(x^2+y^2-z^2\}, z^2-2xy) = 0$.

Sol:- Given that $f(x^2+y^2-z^2, z^2-2xy) = 0$

Which is in implicit form, where f is an arbitrary function.

Its explicit form can be written as $x^2+y^2+z^2 = \phi(z^2-2xy)$ —(1).

Diff (1) w.r.t 'x' partially, we get.

$$2x + 0 - 2z\frac{\partial z}{\partial x} = \phi'(z^2-2xy)\left[2z\frac{\partial z}{\partial x} - 2y\right]$$

$$x - zp = \phi'(z^2-2xy)(zp-y) \quad \text{—(2)}$$

Diff (1) w.r.t 'y' partially, we get

$$0 + 2y - 2z\frac{\partial z}{\partial y} = \phi'(z^2-2xy)\left[2z\frac{\partial z}{\partial y} - 2x\right]$$

$$y - zq = \phi'(z^2-2xy)(zq-x) \quad \text{—(3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{x-zp}{y-zq} = \frac{\phi'(z^2-2xy)(zp-y)}{\phi'(z^2-2xy)(zq-x)}$$

$$(x-zp)(zq-x) = (y-zq)(zp-y)$$

$$pxz + xzq - x^2 - z^2pq = yzp - y^2 - z^2pq + yzq$$

$$pxz + xzq - x^2 = yz(p+q) - y^2$$

$$xz(p+q) - x^2 = yz(p+q) - y^2$$

$$xz(p+q) - yz(p+q) = x^2 - y^2$$

$$z(p+q)(x-y) = (x+y)(x-y)$$

$$z(p+q) = x+y$$

Form the partial differential equation by eliminating the arbitrary function from $xy+yz+zx = f\left(\frac{z}{x+y}\right)$.

General Method of solving Equations of first order But of any Degree : Charpit's Methods

This method can be used to solve any first order partial differential equation. We prefer to use charpit's method only when it is not possible to use any of the special methods which we discussed earlier.

Working procedure of charpit's Method :—

Step 1 :- Write the given equation as $f=0$.

Step 2 :- Write charpit's auxiliary equations.

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}.$$

Step 3 :- substitute the partial derivatives in the above auxiliary equations and simplify.

Step 4 :- choose two equations so that the resulting one on integration is a simple relation containing atleast one of p and q.

Step 5 :- solve the one obtained above with the given equation for p and q.

Step 6 :- Substitute these values of p and q in $dz = p\,dx + q\,dy$.

Step 7 :- Integrate the above equation to get the complete integral.

Find the complete integral of $q = (z+px)^2$ using Charpit's method.

Sol:- Given that $q = (z+px)^2$.

Let $f(x,y,z,p,q) = (z+px)^2 - q = 0$. —— ①

$f_x = 2(z+px)p$. $\quad f_y = 0 \quad f_z = 2(z+px)$

$f_p = 2(z+px)x \quad f_q = -1$.

Charpit's Auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$f_x + pf_z = 4p(z+px)$

$f_y + qf_z = 2q(z+px)$

$-pf_p - qf_q = -2px(z+px) + q$

Sub. all these in above auxiliary equations, we get

$$\frac{dp}{4p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2px(z+px)+q} = \frac{dx}{-2x(z+px)} = \frac{dy}{1}$$

Taking 2nd and 4th members, we get

$$\frac{dq}{2q(z+px)} = \frac{dx}{-2x(z+px)}$$

Integrating both sides, we get

$$\int \frac{dq}{q} = -\int \frac{dx}{x} + \log c.$$

$$\log|q| = -\log|x| + \log c.$$

$$\log|q| = \log\left|\frac{c}{x}\right|$$

$$q = \frac{c}{x} \quad . —— ②$$

sub: ② in ①, we get

$$(z+px)^2 - \frac{a}{x} = 0.$$

$$(z+px)^2 = \frac{a}{x}.$$

$$z+px = \sqrt{\frac{a}{x}}.$$

$$px = \sqrt{\frac{a}{x}} - z$$

$$p = \frac{\sqrt{a}}{x^{3/2}} - \frac{z}{x}.$$

We have $$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{a}\, x^{-3/2} dx - \frac{z}{x} dx + \frac{a}{x} dy.$$

$$dz = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right) dx + \frac{a}{x} dy$$

$$x\,dz = \sqrt{a}\, x^{-1/2} dx - z\,dx + a\,dy.$$

$$x\,dz + z\,dx = \sqrt{a}\, x^{-1/2} dx + a\,dy.$$

$$d(xz) = \sqrt{a}\, x^{-1/2} dx + a\,dy.$$

Integrating bothsides, we get

$$\int d(xz) = \sqrt{a} \int x^{-1/2} dx + a \int dy + c.$$

$$xz = 2\sqrt{a}\sqrt{x} + ay + c.$$

$$xz = 2\sqrt{ax} + ay + c.$$

Which is the complete sol. of given P.D.E

Find a complete, singular and general integrals of $(p^2+q^2)y = qz$.

Sol:- Given that $(p^2+q^2)y = qz$

Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ —— ①.

$f_x = 0 \qquad f_y = p^2+q^2 \qquad f_z = -q.$

$f_p = 2py \qquad f_q = 2qy - z$

Charpits Auxiliary equations are.

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$f_x + pf_z = -pq.$

$f_y + qf_z = p^2+q^2-q^2 = p^2$

$-pf_p - qf_q = -2p^2y - 2q^2y + qz$

Sub. all these in above auxiliary equations, we get.

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{z - 2qy}$$

Taking the first two members, we get

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$

$$pdp + qdq = 0$$

Integrating both sides, we get

$$\int pdp + \int qdq = c_1$$

$$\frac{p^2}{2} + \frac{q^2}{2} = c_1$$

$$p^2 + q^2 = 2c_1$$

$p^2+q^2 = a^2$ where $a^2 = 2c_1$. —— ②

Sub. ② in ①, we get

$$a^2y - qz = 0$$

$$\Rightarrow q = \frac{a^2y}{z} \text{ —— ③}$$

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Sub. (3) in (2), we get.

$$p^2 + \frac{a^4y^2}{z^2} = a^2$$

$$p^2 = a^2 - \frac{a^4y^2}{z^2}$$

$$p^2 = \frac{a}{z}\sqrt{z^2-a^2y^2}$$

We have $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}dy.$$

$$z\,dz = a\sqrt{z^2-a^2y^2}\,dx + a^2y\,dy$$

$$z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dx$$

$$\frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx.$$

Integrating bothsides, we get

$$\frac{1}{2}\int \frac{2z\,dz - a^2 2y\,dy}{\sqrt{z^2-a^2y^2}} = a\int dx + c$$

$$\frac{1}{2}\cdot 2\sqrt{z^2-a^2y^2} = ax + c.$$

$$\sqrt{z^2-a^2y^2} = ax+c.$$

$$z^2 - a^2y^2 = (ax+c)^2 \quad \text{——— (4)}.$$

Which is a required complete integral. a, b being arbitrary constants.

Singular Integral :—

Diff (4) w.r.t a and b partially, we get

$$0 - 2ay^2 = 2(ax+c)x.$$

$$2ay^2 + 2x(ax+c) \quad \text{——— (5)}$$

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$$0 = 2(ax+b) \quad \text{—— (6)}$$

Eliminating a and c between (4), (5) and (6), we get $z=0$.
Which is clearly satisfies (1) and hence it is the singular integral.

General Integral :-

Replace c by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2 \quad \text{—— (7)}.$$

Diff (7) w.r.t 'a' partially, we get.

$$-2ay^2 = 2[ax + \phi(a)][x + \phi'(a)] \quad \text{—— (8)}.$$

General integral is obtained by eliminating a from (7) and (8).

2. Improper Integrals

Introduction

consider the integral $\int_a^b f(x)dx$. such an integral, for which

i) Either the interval of integration is not finite i.e.,

$a = -\infty$ (or) $b = \infty$ (or) both

ii) or the function $f(x)$ is unbounded at one or more points in the closed interval $[a,b]$ is called an improper integral.

iii) Integral corresponding to (i) and (ii) are called improper integrals of the first and second kind respectively. Integrals which satisfy both the conditions (i) and (ii) are called improper integrals of third type.

For example.

1) $\int_0^\infty \frac{dx}{1+x^4}$ and $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ are improper integrals of (I) kind.

2) $\int_0^1 \frac{1}{1-x^2} dx$ is an improper integral of (II) kind.

3) The gamma function defined by the integral

$\int_0^\infty e^{-x} x^{n-1} dx$ when $n>0$ is an improper integral of the third kind.

Beta function:-

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ is called the beta function. and is denoted by $B(m,n)$ (or) $\beta(m,n)$ and read as Beta of (m,n) i.e.,

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$

The above integral converges for $m>0, n>0$

Beta function is also called Eulerian integral of the first kind

Properties of Beta function

1) Symmetry of the Beta function i.e., $\beta(m,n)=\beta(n,m)$

Proof

By the definition of Beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{--- (1)}$$

put $1-x=y$

$-dx=dy$

$dx=-dy$

and $x=0 \Rightarrow y=1$

$x=1 \Rightarrow y=0$

$$(1) \Rightarrow B(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1}(-dy)$$

$$= -\int_1^0 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$= B(n,m)$ [$\because$ By the def^n of Beta function]

$\therefore B(m,n)=B(n,m)$

2) $B(m,n) = 2\int_{0}^{\frac{\pi}{2}} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta$

Proof

By the definition of Beta function.

$$B(m,n) = \int_{0}^{1} x^{m-1}(1-x)^{n-1}\,dx \quad — ①$$

put $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

and $x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$① \Rightarrow B(m,n) = \int_{0}^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}(2\sin\theta\cos\theta\,d\theta)$$

$$= \int_{0}^{\frac{\pi}{2}} \sin^{2m-2}\theta \cdot (\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_{0}^{\frac{\pi}{2}} \sin^{2m-2}\theta \ \cos^{2n-2}\theta \ \sin\theta\cos\theta\,d\theta$$

$$= 2\int_{0}^{\frac{\pi}{2}} \sin^{2m-1}\theta \ \cos^{2m-1}\theta\,d\theta$$

$$\boxed{B(m,n) = 2\int_{0}^{\frac{\pi}{2}} \sin^{2m-1}\theta \ \cos^{2n-1}\theta\,d\theta}$$

Note: $\int_{0}^{\frac{\pi}{2}} \sin^{2m-1}\theta \ \cos^{2n-1}\theta\,d\theta = \frac{1}{2}B(m,n)$

3) $B(m,n) = B(m+1,n) + B(m,n+1)$

Proof

By the def^n of beta function

$$B(m,n) = \int_{0}^{1} x^{m-1}(1-x)^{n-1}\,dx \quad — ①$$

R.H.S: $B(m+1,n)+B(m,n+1) = \int_0^1 x^{(m+1)-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{(n+1)-1}dx$ [$\because$ eqⁿ ①]

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 \left(x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right)dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x+(1-x)]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$= B(m,n)$

$=$ L.H.S

L.H.S = R.H.S

$\therefore B(m,n) = B(m+1,n)+B(m,n+1)$

4) Note:-

If m and n are positive integers, then $B(m,n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

Other forms of Beta functions

Form - I :- To show that

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx \quad (\text{or}) \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

proof

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{--- ①}$$

put $x = \frac{1}{1+y}$

$$dx = \frac{-1}{(1+y)^2}\,dy$$

and $x = 0 \Rightarrow 0 = \frac{1}{1+y} \Rightarrow y = \infty$

$x = 1 \Rightarrow 1 = \frac{1}{1+y} \Rightarrow y = 0$

$y = \infty$ to $y = 0$

$$① \Rightarrow B(m,n) = \int_{y=\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$= -\int_{\infty}^{0} \frac{1}{(1+y)^{m-1}}\left(\frac{1+y-1}{1+y}\right)^{n-1}\left(\frac{1}{(1+y)^2}\right)dy$$

$$= \int_0^{\infty} \frac{1}{(1+y)^{m-1}}\,\frac{y^{n-1}}{(1+y)^{n-1}}\,\frac{1}{(1+y)^2}\,dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$\boxed{\therefore\ B(m,n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy}$$

(or)

$$\boxed{B(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx} \quad \left[\because \int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$$

Again since beta function is symmetrical in m and n and we also have

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Hence $$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(or)

$$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

FORM-ii

Show that $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

PROOF: $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Put $x = 1/y$ $\Rightarrow$ $x=1$ $y=1$

$dx = -1/y^2\,dy$ $\quad$ $x=\infty$ $y=0$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx + \int_1^0 \frac{\left(\frac{1}{y}\right)^{n-1}}{\left[1+\frac{1}{y}\right]^{m+n}} \frac{-dy}{\left[\frac{1}{y}\right]^2}$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{y^{m+n}}{y^{n-1}(1+y)^{m+n}} \frac{dy}{y^2} = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{y^{m+\cancel{n}-\cancel{n}-1}}{(1+y)^{m+n}}dy$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{m-1}}{(1+y)^{m+n}}dy$$

$$B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}$$

FORM-iii

Show that $B(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}dx$

Put $ax = by$

$x = \frac{b}{a}y \qquad x=0 \to y=0$

$dx = \frac{b}{a}dy \qquad x=\infty \to y=\infty$

$$a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}dx = a^m b^n \int_0^\infty \frac{b^{m-1}y^{m-1}}{a^{m-1}b^{m+n}(y+1)^{m+n}} \frac{b}{a}dy$$

$$= a^m b^n \int_0^\infty \frac{b^m . b^{-1} y^{m-1} . b\, dy}{a^m a^{-1} b^m b^n (y+1)^{m+n}} \frac{1}{a}dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}dy = \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}}dx$$

We know that

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= B(m,n)$$

$$\therefore B(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}dx$$

PROVE THAT $\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m,n)}{2a^m b^n}$

WKT $B(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}dx$

Put $x = \tan^2\theta \Rightarrow dx = 2\tan\theta\sec^2\theta\, d\theta$

$x \to 0 \quad \theta = 0$

$x \to \infty \quad \theta = \pi/2$

$$B(m,n) = a^m b^n \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1}\, 2\tan\theta\sec^2\theta\, d\theta}{(a\tan^2\theta + b)^{m+n}}$$

$$= a^m b^n \int_0^{\pi/2} \frac{\tan^{2m-2}\theta\; 2\tan\theta\sec^2\theta\, d\theta}{\left[\frac{a\sin^2\theta + b\cos^2\theta}{\cos^2\theta}\right]^{m+n}}$$

$$= a^m b^n \int_0^{\pi/2} \frac{\sin^{2m-1}\theta\; 2\cos^{2m+2n}\theta \quad d\theta}{\cos^{2m-1}\theta\cos^2\theta\,[a\sin^2\theta + b\cos^2\theta]^{m+n}}$$

$$= 2a^m b^n \int_0^{\pi/2} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\; d\theta}{[a\sin^2\theta + b\cos^2\theta]^{m+n}}$$

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m,n)}{2a^m b^n}$$

FORM-iv

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{B(m,n)}{a^n(1+a)^m}$$

put $x = \dfrac{(1+a)y}{y+a}$

$$dx = 1+a\left[\frac{(y+a)dy - y\,dy}{(y+a)^2}\right]$$

$$dx = \frac{a(1+a)dy}{(y+a)^2} \qquad \begin{matrix} x=0;\ y=0 \\ x=1;\ y=1 \end{matrix}$$

$$B(m,n) = \int_0^1 \left[\frac{(1+a)y}{y+a}\right]^{m-1}\left[1 - \frac{(1+a)y}{y+a}\right]^{n-1} \frac{a(1+a)}{(y+a)^2}dy$$

$$= \int_0^1 \frac{(1+a)^{m-1}y^{m-1}}{(y+a)^{m-1}} \frac{a^{n-1}(1-y)^{n-1}}{(y+a)^{n-1}} \frac{1}{(y+a)^2}\,dy$$

$$= a(1+a)(1+a)^{m-1}a^{n-1}\int_0^1 \frac{y^{m-1}(1-y)^{n-1}}{(y+a)^{m+n}}\,dy$$

$$B(m,n) = a^n(1+a)^m\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx$$

$$\Rightarrow \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{B(m,n)}{a^n(1+a)^m}$$

FORM-V

$$\int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}B(m,n)$$

WKT B function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$x = \frac{y-b}{a-b} \Rightarrow dx = \frac{dy}{a-b}$$

$x=0 \rightarrow y=b$

$x=1 \rightarrow y=a$

$$B(m,n) = \int_b^a \left[\frac{y-b}{a-b}\right]^{m-1}\left[1-\frac{y-b}{a-b}\right]^{n-1}\frac{dy}{a-b}$$

$$= \int_b^a \frac{(y-b)^{m-1}(a-y)^{n-1}}{(a-b)^{m+n-1}}\,dy$$

$$B(m,n) = \int_b^a \frac{(y-b)^{m-1}(a-y)^{n-1}}{(a-b)^{m+n-1}}\,dy$$

$$(a-b)^{m+n-1}B(m,n) = \int_b^a (x-b)^{m-1}(a-x)^{n-1}dx$$

problems:-

(1) $\int_{-1}^{1} [1+x]^{m-1} [1-x]^{n-1} dx = 2^{m+n-1} B(m,n)$

put $x = \frac{y+1}{2} \Rightarrow dx = \frac{dy}{2}$

$x=0 \cdot y=-1$

$x=1 \quad y=1$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$= \int_{-1}^{1} \frac{(y+1)^{m-1}(1-y)^{n-1}}{2^{m+n-1}} dy$$

$$B(m,n) = \frac{1}{2^{m+n-1}} \int_{-1}^{1} (y+1)^{m-1}(1-y)^{n-1} dy$$

$$= \frac{1}{2^{m+n-1}} \int_{-1}^{1} (x+1)^{m-1}(1-x)^{n-1} dx$$

$$2^{m+n-1} B(m,n) = \int_{-1}^{1} (x+1)^{m-1}(1-x)^{n-1} dx$$

Type 1: Express the following integral interms of B function

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$$

& $\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$

put $x^2 = a^2 y$

$x = a\sqrt{y}$

$dx = a \cdot \frac{1}{2\sqrt{y}} dy$

$$= \int_0^1 \frac{1}{\sqrt{a^2-a^2y^2}} \frac{a}{2\sqrt{y}} dy$$

$$= \int_0^1 \frac{1}{a\sqrt{1-y}} \frac{a}{2\sqrt{y}} dy$$

$$= \frac{1}{2}\int_0^1 \frac{dy}{\sqrt{y}\sqrt{1-y}}$$

$$= \frac{1}{2}\int_0^1 y^{-1/2}(1-y)^{-1/2}\,dy$$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{2}\int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

$$= \frac{1}{2}B(m,n)$$

<u>TYPE-II</u> $\int_0^a (a-x)^{m-1}x^{n-1}\,dx = a^{m+n-1}B(m,n)$

Put $x = ay \Rightarrow dx = a\,dy$

$x=0 \quad y=0$

$x=a \quad y=1$

$$\therefore \int_0^a (a-x)^{m-1}x^{n-1}\,dx = \int_0^1 (a-ay)^{m-1}(ay)^{n-1}a\,dy$$

$$= \int_0^1 a^{m-1}(1-y)^{m-1}a^{n-1}y^{n-1}a\,dy$$

$$= \int_0^1 a^{m+n-1}(1-y)^{m-1}\cdot y^{n-1}\,dy$$

$$= a^{m+n-1}\int_0^1 (1-y)^{m-1}y^{n-1}\,dy$$

$$= a^{m+n-1}\int_0^1 (1-x)^{m-1}x^{n-1}\,dx$$

$\left[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\right]$

$$\boxed{B(m,n) = a^{m+n-1}B(n,m)}$$

1) Show that $\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

Soln: Given $\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta$

We know that

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{1}{2}B(m,n) \quad \text{--- (1)}$$

Put $p = 2m-1 \Rightarrow 2m = 1+p \Rightarrow m = \frac{1}{2}(1+p)$

$q = 2n-1 \Rightarrow 2n = 1+q \Rightarrow n = \frac{1}{2}(1+q)$

$$(1) \Rightarrow \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}B\left(\frac{1}{2}(p+1), \frac{1}{2}(q+1)\right)$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

(or) $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

2) Express the following integrals in terms of Beta function

i) $\int_0^1 \frac{x}{\sqrt{1-x^2}} dx$ $\quad (x^2 = y)$

(ii) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ $\quad (x^2 = 9y)$

(iii) $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}}$

(iii) soln: $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}}$

put $x^5 = y \Rightarrow x = y^{1/5}$

$\Rightarrow 5x^4 dx = dy$

$\Rightarrow x^2 dx = \frac{1}{5x^2} dy$

$\Rightarrow x^2 dx = \frac{1}{5y^{2/5}} dy$

and $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$$\int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}} = \int_0^1 \frac{1}{\sqrt{1-y}} \frac{1}{5y^{2/5}} dy$$

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$$= \frac{1}{5}\int_0^1 y^{-2/5}(1-y)^{-1/2}\,dy$$

$$= \frac{1}{5}\int_0^1 y^{\left(\frac{3}{5}-1\right)}(1-y)^{\left(\frac{1}{2}-1\right)}\,dy$$

$$= \frac{1}{5} B\left(\frac{3}{5},\frac{1}{2}\right) \quad \left[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\right]$$

$$\therefore \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^5}} = \frac{1}{5} B\left(\frac{3}{5},\frac{1}{2}\right)$$

(i) solⁿ: $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

Put $x^2 = y \Rightarrow x = y^{1/2}$

$2x\,dx = dy$

$x\,dx = \frac{dy}{2}$

and $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{dy}{2\sqrt{1-y}}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-y}}\,dy$$

$$= \frac{1}{2}\int_0^1 y^0 (1-y)^{-1/2}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{(1-1)}(1-y)^{\left(\frac{1}{2}-1\right)}\,dy$$

$$= \frac{1}{2} B\left(1,\frac{1}{2}\right) \quad \left[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\right]$$

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{1}{2} B\left(1, \frac{1}{2}\right)$$

(ii) sol^n: $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

put $x^2 = 9y \Rightarrow x = 3y^{1/2}$

$2x\,dx = 9\,dy \Rightarrow dx = \frac{9}{2x}dy$

and $x = 0 \Rightarrow y = 0$

$x = 3 \Rightarrow y = 1$

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \int_0^1 \frac{1}{\sqrt{9-9y}} \cdot \frac{9}{2\cdot 3y^{1/2}}\,dy$$

$$= \int_0^1 \frac{1}{\cancel{3}\sqrt{1-y}} \cdot \frac{\cancel{9}}{2\cdot \cancel{3}y^{1/2}}\,dy$$

$$= \frac{1}{2}\int_0^1 \frac{1}{y^{1/2}} \frac{1}{\sqrt{1-y}}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{-1/2}(1-y)^{-1/2}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{\left(\frac{1}{2}-1\right)}(1-y)^{\left(\frac{1}{2}-1\right)}\,dy$$

$$= \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{2}\right) \quad \left[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\right]$$

$$\therefore \int_0^3 \frac{dx}{\sqrt{9-x^2}} = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{2}\right)$$

3) show that $\int_{-1}^{1} (1+x)^{m-1} (1-x)^{n-1} dx = 2^{m+n-1} B(m,n)$

[put $x = \frac{t-b}{a-b} \Rightarrow \frac{t+1}{1+1} = \frac{t+1}{2}$] by form V

4) show that $\int_{a}^{b} (x-a)^{m} (b-x)^{n} dx = (b-a)^{m+n+1} B(m+1, n+1)$

[put $x = \frac{t-a}{b-a}$ by form V]

5) show that $\int_{5}^{7} (x-5)^{6} (7-x)^{3} dx = 2^{10} B(7,4)$

Soln: By the definition of beta function we have

$$B(m,n) = \int_{0}^{1} x^{m-1} (1-x)^{n-1} dx \quad \text{---} ①$$

put $x = \frac{t-b}{a-b}$

$dx = \frac{1}{a-b} dt$

and $x = 0 \Rightarrow t = b$

$x = 1 \Rightarrow t = a$

$$① \Rightarrow B(m,n) = \int_{b}^{a} \left(\frac{t-b}{a-b}\right)^{m-1} \left(1 - \frac{t-b}{a-b}\right)^{n-1} \cdot \frac{1}{a-b} dt$$

$$= \int_{b}^{a} \frac{(t-b)^{m-1}}{(a-b)^{m-1}} \frac{(a-t)^{n-1}}{(a-b)^{n-1}} \cdot \frac{1}{a-b} dt$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_{b}^{a} (t-b)^{m-1} (a-t)^{n-1} dt$$

$$B(m,n) = \frac{1}{(a-b)^{m+n-1}} \int_{b}^{a} (x-b)^{m-1} (a-x)^{n-1} dx$$

$\left[\because \int_{a}^{b} f(t) dt = \int_{a}^{b} f(x) dx \right]$

$$\Rightarrow B(m,n)(a-b)^{m+n-1} = \int_b^a (x-b)^{m-1}(a-x)^{n-1}dx$$

$$\therefore \int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1} B(m,n)$$

putting $b=5$, $a=7$, $m=7$, $n=4$ we get

$$\therefore \int_5^7 (x-5)^6(7-x)^3 dx = (7-5)^{(7+4-1)} B(7,4)$$

$$= 2^{10} B(7,4)$$

6) show that $\int_0^1 x^m(1-x^n)^p dx = \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$

soln: $\int_0^1 x^m(1-x^n)^p dx$

put $x^n = y \Rightarrow x = y^{1/n}$

$nx^{n-1}dx = dy$

$dx = \frac{dy}{nx^{n-1}}$

$dx = \frac{1}{n \cdot y^{\frac{1}{n}(n-1)}} dy$

and $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$$\therefore \int_0^1 x^m(1-x^n)^p dx = \int_0^1 y^{m/n}(1-y)^p \cdot \frac{1}{n \cdot y^{\frac{n-1}{n}}} dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m}{n}} \cdot y^{\frac{1-n}{n}} (1-y)^p dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}}(1-y)^p\,dy$$

$$= \frac{1}{n}\int_0^1 y^{\left(\frac{m+1}{n}-1\right)}(1-y)^{(p+1)-1}\,dy$$

$$= \frac{1}{n}\int_0^1 x^{\left(\frac{m+1}{n}-1\right)}(1-x)^{(p+1)-1}\,dx \quad \left[\because \int_a^b f(t)dt = \int_a^b f(x)dx\right]$$

$$= \frac{1}{n}B\left(\frac{m+1}{n}, p+1\right)$$

$$\therefore \int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}B\left(\frac{m+1}{n}, p+1\right)$$

1) Prove that $\int_0^a (a-x)^{m-1}x^{n-1}\,dx = a^{m+n-1}B(m,n)$.

Sol^n: $\int_0^a (a-x)^{m-1}x^{n-1}\,dx$

Put $x = ay$

$dx = a\,dy$

and $x=0 \Rightarrow y=0$

$x=a \Rightarrow y=1$

$$\therefore \int_0^a (a-x)^{m-1}x^{n-1}\,dx = \int_0^1 (a-ay)^{m-1}(ay)^{n-1}\,a\,dy$$

$$= \int_0^1 (a(1-y))^{m-1}a^{n-1}y^{n-1}\,a\,dy$$

$$= \int_0^1 a^{m-1}(1-y)^{m-1}a^n y^{n-1}\,dy$$

$$= \int_0^1 a^{m+n-1} (1-y)^{m-1} y^{n-1} \, dy$$

$$= a^{m+n-1} \int_0^1 y^{n-1} (1-y)^{m-1} \, dy$$

$$= a^{m+n-1} \, B(n,m)$$

Since $B(n,m) = B(m,n)$

$$\therefore \int_0^a (a-x)^{m-1} x^{n-1} \, dx = a^{m+n-1} \, B(m,n)$$

Gamma function

The definite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called Gamma function and is denoted by $\Gamma(n)$, we read as gamma of n. i.e.,

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}, \text{ where } n>0$$

gamma function is also called eulerian integral of the second kind.

The integral converges only for $n>0$ and integral doesnot converge if $n \leq 0$

Properties of Gamma function

1) $\Gamma(1) = 1$

2) $\Gamma(n) = (n-1)\Gamma(n-1)$, where $n>1$

i) $\Gamma(n+1) = n\Gamma(n)$

3) If n is a positive fraction then we can write

$\Gamma(n) = (n-1)(n-2)(n-3) \ldots (n-r)\Gamma(n-r)$, where $n-r>0$

eg:- $\Gamma\left(\frac{9}{2}\right) = \left(\frac{9}{2}-1\right)\left(\frac{9}{2}-2\right)\left(\frac{9}{2}-3\right)\left(\frac{9}{2}-4\right)\Gamma\left(\frac{9}{2}-4\right)$, $\left(\frac{9}{2}-4\right)>0$

$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$

4) $\Gamma(n) = (n-1)!$, where n is a non-negative integer.

$\Gamma(n+1) = n!$

Ex:- $\Gamma(4) = 3! = 6$

Other forms of Gamma functions

Form-1:

To prove that $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

Soln: By the definition of gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Put $x = ky$

$dx = k\,dy$

and $x = 0 \Rightarrow y = 0$

$x \to \infty \Rightarrow y \to \infty$

$$\therefore \Gamma(n) = \int_0^\infty e^{-ky} (ky)^{n-1} k\,dy$$

$$= \int_0^\infty e^{-ky} k^{n-1+1} y^{n-1} dy$$

$$= \int_0^\infty e^{-ky} k^n y^{n-1} dy$$

$$\Gamma(n) = k^n \int_0^\infty e^{-ky} \cdot y^{n-1} dy$$

$$\Gamma(n) = k^n \int_0^\infty e^{-kx} x^{n-1} dx \quad \left[\because \int_a^b f(t)dt = \int_a^b f(x)dx\right]$$

$$\boxed{\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}}$$

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Form-2:

To prove that $\gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}} dx$

Proof: By the definition of gamma function

$$\gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

put $x^n = y \Rightarrow x = y^{1/n}$

$$nx^{n-1} dx = dy \Rightarrow dx = \frac{1}{nx^{n-1}} dy$$

$$\left[dx = \frac{1}{\frac{x}{n}\cdot y}\right] \quad x^{n-1} dx = \frac{1}{n} dy$$

and $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y \to \infty$

$$\gamma(n) = \int_0^\infty e^{-y^{1/n}} \frac{1}{n} dy$$

$$= \frac{1}{n}\int_0^\infty e^{-y^{1/n}} dy$$

$$\gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}} dx \quad \left[\because \int_a^b f(t)dt = \int_a^b f(x)dx\right]$$

Form-3:

To prove that $\gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

Proof:

By the definition of gamma function

$$\gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

put $e^{-x} = y \Rightarrow x = \log\frac{1}{y}$

$$-e^{-x}dx = dy$$

$$e^{-x}dx = -dy$$

and $x=0 \Rightarrow y = 1$

$x \to \infty \Rightarrow y \to 0$

$$\Gamma(n) = \int_1^0 \left(\log\frac{1}{y}\right)^{n-1}(-dy)$$

$$\Gamma(n) = -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} dy$$

$$\boxed{\Gamma(n) = \int_0^1 \left(\log\left(\frac{1}{y}\right)\right)^{n-1} dy}$$

Form-4:

To prove that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$

Proof:

By the definition of gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

put $x = y^2$ and $x=0 \Rightarrow y=0$

$dx = 2y\,dy$ $\quad x=\infty \Rightarrow y=\infty$

$$\Gamma(n) = \int_0^\infty e^{-y^2}(y^2)^{n-1}\, 2y\,dy$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-2}\, y\, dy$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$\boxed{\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx} \quad \left[\because \int_a^b f(t)dt = \int_a^b f(x)dx\right]$$

Problems

1) Evaluate i) $\int_0^1 x^4 (\log \frac{1}{x})^3 dx$

ii) $\int_0^1 x^2 (\log \frac{1}{x})^3 dx$

Sol^n: i) put $\log \frac{1}{x} = y \Rightarrow x = e^{-y}$

$$dx = -e^{-y} dy$$

$x = 0 \Rightarrow y = \infty$

$x = 1 \Rightarrow y = 0$

$$\therefore \int_0^1 x^4 (\log \tfrac{1}{x})^3 dx = \int_\infty^0 e^{-4y} y^3 (-e^{-y} dy)$$

$$= -\int_\infty^0 e^{-5y} y^3 dy$$

$$= \int_0^\infty e^{-5y} y^3 dy$$

Put $5y = t$

$\Rightarrow 5dy = dt$

$dy = \frac{dt}{5}$

and $y = 0 \Rightarrow t = 0$

$y \to \infty \Rightarrow t \to \infty$

$$\int_0^1 x^4 (\log \tfrac{1}{x})^3 dx = \int_0^\infty e^{-5y} y^3 dy = \int_0^\infty e^{-t} \left(\frac{t}{5}\right)^3 \cdot \frac{dt}{5}$$

$$= \frac{1}{5^4} \int_0^\infty e^{-t} t^{(4-1)} dt$$

$$= \frac{1}{5^4} \Gamma(4)$$

$= \frac{3!}{5^4} = \frac{6}{625}$

(or)

$$\int_0^1 x^4 (\log\tfrac{1}{x})^3 dx = \int_0^\infty e^{-5y} y^3 dy$$

By form 1

we have $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\gamma(n)}{k^n}$

$$\Rightarrow \int_0^\infty e^{-5y} y^3 dy = \int_0^\infty e^{-5y} y^{(4-1)} dy$$

$k = 5$, $n = 4$

$$\therefore \int_0^\infty e^{-5y} y^3 dy = \frac{\gamma(4)}{5^4} = \frac{3!}{5^4} = \frac{6}{625}$$

ii) B. $\int_0^1 x^2 (\log(\frac{1}{x}))^3 dx$

put $\log\frac{1}{x} = y \Rightarrow x = e^{-y}$

$dx = -e^{-y} dy$

$x = 0 \Rightarrow y = \infty$

$x = 1 \Rightarrow y = 0$

$$\therefore \int_0^1 x^2 (\log\tfrac{1}{x})^3 dx = \int_\infty^0 e^{-2y} y^3 (-e^{-y} dy)$$

$$= -\int_\infty^0 e^{-3y} y^3 dy$$

$$= \int_0^\infty e^{-3y} y^3 dy = \int_0^\infty e^{-3y} y^{(4-1)} dy$$

$k = 3$, $n = 4$

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By form 1

$$\therefore \int_0^{\infty} e^{-3y} y^3 dy = \frac{\Gamma(4)}{3^4} = \frac{3!}{3^4} = \frac{6}{81} = \frac{2}{27}$$

Beta functions

3 soln: $\int_{-1}^{1} (1+x)^{m-1} (1-x)^{n-1} dx = 2^{m+n-1} B(m,n)$

By the definition of beta function we have

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \text{--- (1)}$$

put $x = \frac{t-b}{a-b}$

Here by the form V

$$\int_b^a (x-b)^{m-1} (a-x)^{n-1} dx = (a-b)^{m+n-1} B(m,n)$$

by comparing $a=1$, $b=-1$

$$x = \frac{t-b}{a-b} = \frac{t+1}{1+1}$$

put $x = \frac{t+1}{2} = \frac{1}{2}(t+1)$

$$dx = \frac{1}{2} dt$$

and $\boxed{\begin{array}{l} x = \cancel{0}\ {-1} \Rightarrow t = -3 \\ x = 1 \Rightarrow t = 1 \end{array}}$ $\quad$ $x = 0 \Rightarrow t = -1$, $x = 1 \Rightarrow t = 1$

$$(1) \Rightarrow B(m,n) = \int_{-1}^{1} \left(\frac{t+1}{2}\right)^{m-1} \left(1 - \frac{t+1}{2}\right)^{n-1} \cdot \frac{1}{2} dt$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\frac{t+1}{2}\right)^{m-1}\left(\frac{2-t-1}{2}\right)^{n-1} dt$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\frac{t+1}{2}\right)^{m-1}\left(\frac{1-t}{2}\right)^{n-1} dt$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{(1+t)^{m-1}}{2^{m-1}}\,\frac{(1-t)^{n-1}}{2^{n-1}}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{(1+t)^{m-1}(1-t)^{n-1}}{2^{m+n-2}}\, dt$$

$$B(m,n) = \frac{1}{2^{m+n-1}}\int_{-1}^{1}(1+t)^{m-1}(1-t)^{n-1}\, dt$$

$$2^{m+n-1}B(m,n) = \int_{-1}^{1}(1+t)^{m-1}(1-t)^{n-1}\, dt$$

$$2^{m+n-1}B(m,n) = \int_{-1}^{1}(1+x)^{m-1}(1-x)^{n-1}\, dx \quad \left[\because \int_a^b f(t)dt = \int_a^b f(x)dx\right]$$

$$\therefore \int_{-1}^{1}(1+x)^{m-1}(1-x)^{n-1}\, dx = 2^{m+n-1}\cdot B(m,n)$$

4sol^n: $\int_a^b (x-a)^m (b-x)^n dx$.

By the definition of beta function we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad \text{—}①$$

put $x = \frac{t-a}{b-a}$

$$dx = \frac{1}{b-a}\, dt$$

and $x=0 \Rightarrow t=a$

$x=1 \Rightarrow t=b$

$$(1) \Rightarrow B(m,n) = \int_a^b \left(\frac{t-a}{b-a}\right)^{m-1}\left(1-\frac{t-a}{b-a}\right)^{n-1} \frac{1}{b-a}\,dt$$

$$= \int_a^b \frac{(t-a)^{m-1}}{(b-a)^{m-1}} \left(\frac{b-\cancel{a}-t+\cancel{a}}{b-a}\right)^{n-1} \frac{1}{b-a}\,dt$$

$$= \int_a^b \frac{(t-a)^{m-1}}{(b-a)^{m-1}} \frac{(b-t)^{n-1}}{(b-a)^{n-1}} \cdot \frac{1}{b-a}\,dt$$

$$= \int_a^b \frac{1}{(b-a)^{m+n-1}} (t-a)^{m-1} (b-t)^{n-1}\,dt$$

$$B(m,n) = \frac{1}{(b-a)^{m+n-1}} \int_a^b (t-a)^{m-1}(b-t)^{n-1}\,dt$$

$$(b-a)^{m+n-1} B(m,n) = \int_a^b (t-a)^{m-1}(b-t)^{n-1}\,dt$$

$$\Rightarrow (b-a)^{m+n-1} \cdot B(m,n) = \int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx \quad \left[\because \int_a^b f(t)dt = \int_a^b f(x)dx\right]$$

Let $m=m+1$, $n=n+1$

$$\Rightarrow (b-a)^{m+1+n+\cancel{1}-\cancel{1}} B(m+1,n+1) = \int_a^b (x-a)^{m+1-1}(b-x)^{n+1-1}\,dx$$

$$\Rightarrow (b-a)^{m+n+1} B(m+1,n+1) = \int_a^b (x-a)^m (b-x)^n\,dx$$

$$\therefore \int_a^b (x-a)^m (b-x)^n\,dx = (b-a)^{m+n+1} B(m+1,n+1)$$

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Relation between beta and gamma functions

The relation between beta and gamma functions is

$$B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

proof

By other form of gamma function

$$\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1}\,dx \quad \text{--- (1)}$$

$$\Gamma(m) = 2\int_0^{\infty} e^{-y^2} y^{2m-1}\,dy \quad \text{--- (2)}$$

Multiplying eq^ns (1) and (2) we get

$$\Gamma(m)\cdot\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2m-1}\,dy \times 2\int_0^{\infty} e^{-x^2} x^{2n-1}\,dx$$

$$= 2.2\int_0^{\infty}\int_0^{\infty} e^{-x^2} e^{-y^2} x^{2n-1} y^{2m-1}\,dx\,dy$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\,dx\,dy \quad \text{--- (3)}$$

The region of integration in the double integral eq^n (3) is the infinite area in the first quadrant, now changing into polar coordinates by putting

$x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$x^2+y^2 = r^2$$

and θ varies from $\theta = 0$ to $\frac{\pi}{2}$ and r varies from 0 to ∞.

$\therefore$ eq^n ③ $\Rightarrow$ $\Gamma(m)\cdot\Gamma(n) = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1}\, r\,dr\,d\theta$

$$= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}\cdot r^{2n-1}\cdot\cos^{2n-1}\theta\; r^{2m-1}\sin^{2m-1}\theta\; r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}\, r^{2m+2n-1}\cos^{2n-1}\theta\,\sin^{2m-1}\theta\, d\theta\, dr$$

$$= \left(2\int_{\theta=0}^{\frac{\pi}{2}} \sin^{2m-1}\cos^{2n-1}\theta\, d\theta\right)\left(2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr\right)$$

$$= \cancel{2}\,\frac{1}{\cancel{2}}\, B(m,n)\left(\Gamma(m+n)\right)$$ [$\because$ By property of beta function & using eq^n ①]

$$\Gamma(m)\cdot\Gamma(n) = B(m,n)\,\Gamma(m+n)$$

$$\therefore \boxed{B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}$$

Problems

1) To show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sol^n: By the relation between beta and gamma functions.

$$B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} \quad \text{—①}$$

putting $m=\frac{1}{2}$, $n=\frac{1}{2}$ in eq^n ①

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\frac{\gamma\left(\tfrac{1}{2}\right)\cdot\gamma\left(\tfrac{1}{2}\right)}{\gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\right)}$$

$$=\frac{\left(\gamma\left(\tfrac{1}{2}\right)\right)^2}{\gamma(1)}$$

$$=\left(\gamma\left(\tfrac{1}{2}\right)\right)^2 \quad [\because \gamma(1)=1]$$

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\left(\gamma\left(\tfrac{1}{2}\right)\right)^2 \quad \text{— } 2$$

By the definition of Beta function

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\int_0^1 x^{\left(\frac{1}{2}-1\right)}(1-x)^{\frac{1}{2}-1}\,dx$$

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

put $x=\sin^2\theta$

$dx=2\sin\theta\cos\theta\,d\theta$

and $x=0 \Rightarrow \theta=0$

$x=1 \Rightarrow \theta=\frac{\pi}{2}$

$$\therefore B\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\int_0^{\pi/2}(\sin^2\theta)^{-1/2}(1-\sin^2\theta)^{-1/2}\,\cancel{dx}\;2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}(\sin^2\theta)^{-1/2}(\cos^2\theta)^{-1/2}\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}\frac{1}{\sin\theta\cos\theta}\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}d\theta=2(\theta)_0^{\pi/2}$$

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$= 2\left[\frac{\pi}{2}-0\right)$

$= \cancel{2}\left[\frac{\pi}{\cancel{2}}\right]$

$= \pi$

$\therefore B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

Now sub $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$ in eqn 2

$2 \Rightarrow \pi = \left(\Gamma\left(\frac{1}{2}\right)\right)^2$

$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2) Evaluate the following integrals:

i) $\int_0^1 x^5(1-x)^3\,dx$ ii) $\int_0^1 x^4(1-x)^2\,dx$

soln: i) $\int_0^1 x^5(1-x)^3\,dx$

$= \int_0^1 x^{(6-1)}(1-x)^{4-1}\,dx$

$= B(6,4)$ $\left[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\right]$

$= \frac{\Gamma(6)\Gamma(4)}{\Gamma(6+4)}$ $\left[\because B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\right]$

$= \frac{5!\,3!}{9!}$ $\left[\because \Gamma(n) = (n-1)!\right]$

$= \frac{\cancel{5!}\,\cancel{3!}}{9\cdot 8\cdot 7\cdot \cancel{6}\cdot \cancel{5!}} = \frac{1}{504}$

$= \frac{1}{504}$

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ii) $\int_0^1 x^4(1-x)^2 dx$

$= \int_0^1 x^{(5-1)}(1-x)^{3-1} dx$

$= B(5,3) \quad [\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx]$

$= \frac{\gamma(5)\gamma(3)}{\gamma(5+3)} \quad [\because B(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}]$

$= \frac{4!\,2!}{7!}$

$= \frac{4!\,2!}{7\cdot 6\cdot 5\cdot 4!} = \frac{\cancel{2}}{\cancel{210}\,105}$

$= \frac{1}{105}$

3) Evaluate the following

i) $\int_0^\infty x^6 e^{-2x} dx$ ii) $\int_0^\infty e^{-x^2} x^2 dx$

Soln: i) $\int_0^\infty x^6 e^{-2x} dx$

put $2x = y \Rightarrow x = y/2$

$2dx = dy \Rightarrow dx = \frac{dy}{2}$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$\int_0^\infty x^6 e^{-2x} dx = \int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y} \cdot \frac{dy}{2}$

$= \frac{1}{2^7}\int_0^\infty e^{-y} y^6 dy$

$$= \frac{1}{2^7}\int_0^\infty e^{-y}\, y^{(7-1)}\,dy$$

$$= \frac{1}{2^7}\Gamma(7) \quad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx\right]$$

$$= \frac{6!}{2^7} \quad \left[\because \Gamma(n) = (n-1)!\right]$$

$$\therefore \int_0^\infty e^{-2x} x^6\,dx = \frac{45}{8}$$

ii) $\int_0^\infty e^{-x^2} x^2\,dx$

put $x^2 = y \Rightarrow x = y^{1/2}$

$2x\,dx = dy \Rightarrow dx = \frac{dy}{2x}$

and $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\int_0^\infty e^{-x^2} x^2\,dx = \int_0^\infty e^{-y} \cdot y \cdot \frac{dy}{2y^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty e^{-y} \cdot y^{1/2}\,dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y} \cdot y^{\left(\frac{3}{2}-1\right)}\,dy$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{4} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

4) Evaluate the following integrals.

i) $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$ ii) $\int_0^{\pi/2} \sin^7\theta \, d\theta$

iii) $\int_0^{\pi/2} \cos^{11}\theta \, d\theta$ iv) $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

Sol^n: i) $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$

By property of beta function

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$$

$\Rightarrow 2m-1 = 5$; $2n-1 = \frac{7}{2}$

$2m = 6$; $2n = \frac{7}{2}+1 = \frac{9}{2}$

$m = 3$

$n = \frac{9}{4}$

$m = 3, \ n = \frac{9}{4}$

$$\therefore \int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta = \frac{1}{2} B\left(3, \frac{9}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(3)\,\Gamma(9/4)}{\Gamma\left(3+\frac{9}{4}\right)} \quad \left[\because B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{2} \cdot \frac{2! \cdot \Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{21}{4}\right)}$$

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\ \Gamma\left(\frac{9}{4}\right)}{\left(\frac{17}{4}\right)\left(\frac{13}{4}\right)\left(\frac{9}{4}\right)\Gamma\left(\frac{9}{4}\right)}$$

[$\because$ if n is the +tve fraction then $\Gamma(n) = (n-1)(n-2)\ldots(n-r)\,\Gamma(n-r)$, $(n-r)>0$]

$$= \frac{1}{\left(\frac{17}{4}\right)\left(\frac{13}{4}\right)\left(\frac{9}{4}\right)} = \frac{64}{1989}$$

iii) $\int_0^{\frac{\pi}{2}} \sin^7\theta \, d\theta$

By the property of beta function

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$$

put $2m-1 = 7$ & $2n-1 = 0$

$2m = 8$, $2n = 1$

$m = 4$ $n = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^7\theta \, d\theta = \frac{1}{2} B\left(4, \frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(4)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(4+\frac{1}{2}\right)} \quad \left[\because B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{2} \cdot \frac{3!\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{3!\,\cancel{\Gamma\left(\frac{1}{2}\right)}}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\cancel{\Gamma\left(\frac{1}{2}\right)}}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{6}^{2} \cdot 8 \cdot 2}{(7)(5)(\cancel{3})}$$

$$= \frac{16}{35}$$

iv) $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta} \, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos\theta}{\sin\theta}} \, d\theta = \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$

By the property of beta function

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$$

Put $2m-1 = -\frac{1}{2}$, $2n-1 = \frac{1}{2}$

$2m = -\frac{1}{2}+1$, $2n = \frac{1}{2}+1$

$2m = \frac{1}{2}$, $2n = \frac{3}{2}$

$m = \frac{1}{4}$, $n = \frac{3}{4}$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}B\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)} \quad \left[\because B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

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$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)}{1}$$

$$= \frac{1}{2} \cdot \Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}} \quad \left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{2} \cdot \frac{\pi}{(1/\sqrt{2})}$$

$$= \frac{\pi\sqrt{2}}{2}$$

$$\frac{\pi\sqrt{2}}{2} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\pi(2)}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}}$$

iii) $\int_0^{\frac{\pi}{2}} \cos^{11}\theta\, d\theta$

By the property of beta function

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}B(m,n)$$

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put $2m-1=0$, $2n-1=11$

$2m=1$, $2n=12$

$m=1/2$, $n=6$

$$\therefore \int_0^{\pi/2} \cos^{11}\theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, 6\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(6)}{\Gamma\left(\frac{1}{2}+6\right)} \qquad \left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right) 5!}{\Gamma\left(\frac{13}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\cancel{\Gamma\left(\frac{1}{2}\right)}\, 5!}{\left(\frac{11}{2}\right)\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\cancel{\Gamma\left(\frac{1}{2}\right)}}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\overset{60}{\cancel{120}} \times 64}{10395} = \frac{3840}{10395}$$

5) Evaluate

i) $\int_0^2 x(8-x^3)^{1/3}\, dx$ ii) $\int_0^1 x^{5/2}(1-x^2)^{3/2}\, dx$ $(x^2=y)$

sol^n: i) $\int_0^2 x(8-x^3)^{1/3}\, dx$

Put $x^3=8y \Rightarrow x=2y^{1/3}$

$3x^2dx = dy \Rightarrow dx = \frac{1}{3}x^{-2}dy = \frac{2}{3}y^{-2/3}dy$

and $x=0 \Rightarrow y=0$

$x=2 \Rightarrow y=1$

$$\int_0^2 x(8-x^3)^{1/3}\, dx = \int_0^1 2y^{1/3}(8-8y)^{1/3} \cdot \frac{2}{3}y^{-2/3}\, dy$$

$$= \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3}\,dy$$

$$= \frac{8}{3}\int_0^1 y^{\left(\frac{2}{3}-1\right)}(1-y)^{\left(\frac{4}{3}-1\right)}\,dy$$

$$= \frac{8}{3}\,B\left(\frac{2}{3},\frac{4}{3}\right) \quad \left[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\right]$$

$$= \frac{8}{3}\cdot\frac{\gamma\left(\frac{2}{3}\right)\cdot\gamma\left(\frac{4}{3}\right)}{\gamma\left(\frac{2}{3}+\frac{4}{3}\right)} \quad \left[\because B(m,n) = \frac{\gamma(m)\cdot\gamma(n)}{\gamma(m+n)}\right]$$

$$= \frac{8}{3}\cdot\frac{\gamma\left(\frac{2}{3}\right)\left(\frac{4}{3}-1\right)\gamma\left(\frac{4}{3}-1\right)}{\gamma(2)}$$

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$$= \frac{8}{3}\cdot\frac{\gamma\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\gamma\left(\frac{1}{3}\right)}{1!}$$

$$= \frac{8}{3}\cdot\frac{1}{3}\,\gamma\left(\frac{2}{3}\right)\gamma\left(\frac{1}{3}\right)$$

$$= \frac{8}{9}\,\gamma\left(\frac{1}{3}\right)\gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{8}{9}\cdot\frac{\pi}{\sin\frac{\pi}{3}} \quad \left[\because \gamma(n)\gamma(n-1) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{8\pi}{9\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{16\pi}{9\sqrt{3}}$$

ii) $\int_0^1 x^{5/2}(1-x^2)^{3/2}\,dx$

put $x^2 = y \Rightarrow x = y^{1/2}$

$2x\,dx = dy \Rightarrow dx = \frac{dy}{2x} = \frac{dy}{2y^{1/2}}$

and $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$$\int_0^1 x^{5/2}(1-x^2)^{3/2}\,dx = \int_0^1 y^{5/4}(1-y)^{3/2}\cdot\frac{dy}{2y^{1/2}}$$

$$= \frac{1}{2}\int_0^1 y^{\frac{5}{4}-\frac{1}{2}}(1-y)^{3/2}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{3/4}(1-y)^{3/2}\,dy$$

$$= \frac{1}{2}\int_0^1 y^{\left(\frac{7}{4}-1\right)}(1-y)^{\left(\frac{5}{2}-1\right)}\,dy$$

$$= \frac{1}{2}B\left(\frac{7}{4},\frac{5}{2}\right) \quad \left[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\right]$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{4}+\frac{5}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\left(\frac{7}{4}-1\right)\Gamma\left(\frac{7}{4}-1\right)\cdot\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)}{\left(\frac{13}{4}\right)\left(\frac{9}{4}\right)\left(\frac{5}{4}\right)\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{1}{\cancel{2}}\,\frac{\left(\frac{\cancel{3}}{\cancel{4}}\right)\Gamma\left(\frac{3}{4}\right)\cdot\left(\frac{\cancel{3}}{\cancel{2}}\right)\Gamma\left(\frac{3}{2}\right)}{\left(\frac{13}{\cancel{4}}\right)\left(\frac{\cancel{9}}{\cancel{4}}\right)\left(\frac{5}{16}\right)\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\frac{(13)(5)}{64}\,\Gamma\left(\frac{1}{4}\right)} = \frac{64}{65}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

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6) Poove that $\int_0^1 (1-x^n)^{1/n}\,dx = \frac{1}{n}\,\frac{(\gamma(\frac{1}{n}))^2}{2\gamma(\frac{2}{n})}$

soln: put $x^n = y \Rightarrow x = y^{1/n}$

$$nx^{n-1}dx = dy$$

$$dx = \frac{1}{nx^{n-1}}dy$$

$$dx = \frac{1}{ny^{\frac{n-1}{n}}}dy$$

and $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$$\therefore \int_0^1 (1-x^n)^{1/n}\,dx = \int_0^1 (1-y)^{1/n}\cdot\frac{1}{ny^{\frac{n-1}{n}}}dy$$

$$= \int_0^1 \frac{1}{n}(1-y)^{1/n}\,y^{-\frac{n+1}{n}}\,dy$$

$$= \frac{1}{n}\int_0^1 y^{(\frac{1}{n}-1)}(1-y)^{(\frac{1}{n}+1)-1}\,dy$$

$$= \frac{1}{n}B\left(\frac{1}{n}, \frac{1}{n}+1\right)$$

$$= \frac{1}{n}B\left(\frac{1}{n}, \frac{n+1}{n}\right)$$

$$= \frac{1}{n}\cdot\frac{\gamma(\frac{1}{n})\cdot\gamma(\frac{n+1}{n})}{\gamma(\frac{1}{n}+\frac{n+1}{n})}$$

$$= \frac{1}{n}\,\frac{\gamma(\frac{1}{n})\,(\frac{n+1}{n}-1)\,\gamma(\frac{n+1}{n}-1)}{\gamma(\frac{n+2}{n})}$$

$$= \frac{1}{n} \cdot \frac{\gamma\left(\frac{1}{n}\right) \cdot \frac{1}{n} \gamma\left(\frac{1}{n}\right)}{\left(\frac{n+2}{n} - 1\right) \gamma\left(\frac{n+2}{n} - 1\right)}$$

$$= \left(\frac{1}{n}\right)^2 \cdot \frac{\left(\gamma(1/n)\right)^2}{\left(\frac{2}{n}\right)\gamma\left(\frac{2}{n}\right)}$$

$$= \frac{1}{n} \cdot \frac{\left(\gamma(1/n)\right)^2}{2\gamma\left(\frac{2}{n}\right)}$$

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7) prove that $\int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx = \frac{\gamma(m)}{n^m}$ $m>0, n>0$

$\log\frac{1}{x} = y$

sol^n: $\int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx$

Put $\log \frac{1}{x} = y \Rightarrow \frac{1}{x} = e^y \Rightarrow x = e^{-y}$

$x \cdot \left(-\frac{1}{x^2}\right) dx = dy$ $\qquad x = 0 \Rightarrow y = \infty$

$\Rightarrow -\frac{1}{x} dx = dy$ $\qquad x = 1 \Rightarrow y = 0$

$\Rightarrow dx = -x\, dy = (-e^{-y}) dy$

$\int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx$

$= \int_\infty^0 (e^{-y})^{n-1} (y)^{m-1} (-e^{-y} dy)$

$= -\int_\infty^0 e^{-(n-1)y - y} y^{m-1} dy$

$= \int_0^\infty e^{(-n)y} y^{m-1} dy$

$$= \int_0^\infty e^{-ny} y^{m-1} dy$$

$$= \frac{\Gamma(m)}{n^m}$$ [∵ From gamma function form-I $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$]

$$\therefore \int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx = \frac{\Gamma(m)}{n^m}$$

8) prove that $\int_0^\infty e^{-x^4} dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$

soln: $\int_0^\infty e^{-x^4} dx$

put $x^4 = y \Rightarrow x = y^{1/4}$

$4x^3 dx = dy$

$$dx = \frac{dy}{4y^{3/4}}$$

$$dx = \frac{1}{4} y^{-3/4} dy$$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\int_0^\infty e^{-x^4} dx = \int_0^\infty e^{-y} \frac{1}{4} y^{-3/4} dy$$

$$= \frac{1}{4} \int_0^\infty e^{-y} y^{\left(\frac{1}{4}-1\right)} dy$$

$$= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$ [∵ $\Gamma(n) = \int_0^\infty e^{-x} x^{(n-1)} dx$]

9) Prove that $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx \times \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\cos x}} = \pi$

Sol^n: Now $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx = \int_0^{\frac{\pi}{2}} \cos^{1/2} x\,dx$

$$= \int_0^{\frac{\pi}{2}} \sin^0 x \cos^{1/2} x\,dx$$

By $\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x\,dx = \frac{1}{2} B(m,n)$

put $2m-1=0$, $2n-1=\frac{1}{2}$

$2m=1$, $2n=\frac{3}{2}$

$m=\frac{1}{2}$, $n=\frac{3}{4}$

$$\int_0^{\frac{\pi}{2}} \sin^0 x \cos^{1/2} x\,dx = \frac{1}{2} B\left(\frac{1}{2}, \frac{3}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{4}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{1}{\cancel{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\frac{1}{\cancel{4}_2}\cdot\Gamma\left(\frac{1}{4}\right)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx = 2\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

Now $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\cos x}} = \int_0^{\frac{\pi}{2}} \cos^{-1/2} x\,dx$

$$= \int_0^{\frac{\pi}{2}} \sin^0 x \cos^{-1/2} x\,dx$$

By $\int_0^{\frac{\pi}{2}} \sin^{2m-1}x \cos^{2n-1}x \, dx = \frac{1}{2} B(m,n)$

put $2m-1=0$, $2n-1=-\frac{1}{2}$

$2m=1$, $2n=\frac{1}{2}$

$m=\frac{1}{2}$, $n=\frac{1}{4}$

$$\int_0^{\frac{\pi}{2}} \cos^{-1/2}x \sin^0 x \, dx = \frac{1}{2}B\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\int_0^{\frac{\pi}{2}} \sin^0 x \cos^{-1/2} x \, dx = \frac{1}{2} \frac{\sqrt{\pi}\, \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx \times \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\cos x}} = \cancel{2} \frac{\sqrt{\pi}\ \cancel{\Gamma\left(\frac{3}{4}\right)}}{\cancel{\Gamma\left(\frac{1}{4}\right)}} \times \frac{1}{\cancel{2}} \frac{\sqrt{\pi}\,\cancel{\Gamma\left(\frac{1}{4}\right)}}{\cancel{\Gamma\left(\frac{3}{4}\right)}}$$

$$= \sqrt{\pi} \times \sqrt{\pi}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx \times \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\cos x}} = \pi$$

10) prove that $\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right]$

soln: $\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta$

$$= \int_0^{\frac{\pi}{2}} \left(\sqrt{\frac{\sin\theta}{\cos\theta}} + \sqrt{\frac{1}{\cos\theta}}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta + \int_0^{\frac{\pi}{2}} \cos^{-1/2}\theta \sin^0\theta \, d\theta \quad \text{--- (1)}$$

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Now $\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$

By $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$

put $2m-1 = \frac{1}{2}$, $2n-1 = -\frac{1}{2}$

$2m = \frac{1}{2}+1$, $2n = -\frac{1}{2}+1$

$2m = \frac{3}{2}$ $2n = \frac{1}{2}$

$m = \frac{3}{4}$, $n = \frac{1}{4}$

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$\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$

$= \frac{1}{2} \frac{\gamma\left(\frac{3}{4}\right) \cdot \gamma\left(\frac{1}{4}\right)}{\gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$

$= \frac{1}{2} \frac{\gamma\left(\frac{3}{4}\right) \cdot \gamma\left(\frac{1}{4}\right)}{\gamma(1)}$

$\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta = \frac{1}{2} \gamma\left(\frac{3}{4}\right) \cdot \gamma\left(\frac{1}{4}\right)$

Now $\int_0^{\frac{\pi}{2}} \sin^{0}\theta \cos^{-1/2}\theta \, d\theta$

By $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$

put $2m-1 = 0$, $2n-1 = -\frac{1}{2}$

$2m = 1$ $2n = -\frac{1}{2}+1$

$m = \frac{1}{2}$

$2n = \frac{1}{2}$

$m = \frac{1}{2}$, $n = \frac{1}{4}$

$\int_0^{\frac{\pi}{2}} \sin^{0}\theta \cos^{-1/2}\theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{4}\right)$

$$= \frac{1}{2}\frac{\gamma\left(\frac{1}{2}\right)\gamma\left(\frac{1}{4}\right)}{\gamma\left(\frac{1}{2}+\frac{1}{4}\right)}$$

$$= \frac{1}{2}\frac{\gamma\left(\frac{1}{2}\right)\gamma\left(\frac{1}{4}\right)}{\gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{2}\frac{\sqrt{\pi}\,\gamma\left(\frac{1}{4}\right)}{\gamma\left(\frac{3}{4}\right)} \qquad \left[\because \gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\int_0^{\frac{\pi}{2}} \sin^0\theta \cos^{-1/2}\theta \, d\theta = \frac{1}{2}\frac{\sqrt{\pi}\,\gamma\left(\frac{1}{4}\right)}{\gamma\left(\frac{3}{4}\right)}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta + \int_0^{\frac{\pi}{2}} \cos^{-1/2}\theta \sin^0\theta \, d\theta$$

$$= \frac{1}{2}\gamma\left(\frac{3}{4}\right)\gamma\left(\frac{1}{4}\right) + \frac{1}{2}\frac{\sqrt{\pi}\,\gamma\left(\frac{1}{4}\right)}{\gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{2}\gamma\left(\frac{1}{4}\right)\left[\gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\gamma\left(\frac{3}{4}\right)}\right]$$

$$\therefore \int_0^{\frac{\pi}{2}} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta = \frac{1}{2}\gamma\left(\frac{1}{4}\right)\left[\gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\gamma\left(\frac{3}{4}\right)}\right]$$

Q Evaluate $\int_0^\infty a^{-bx^2} dx$

Sol put $a = e^{\log_e^a}$

$a^{-bx^2} = \left[e^{\log_e^a}\right]^{-bx^2}$

$a^{-bx^2} = e^{-bx^2 \log_e^a}$

$\therefore \int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-bx^2 \log_e^a} dx$

put $bx^2 \log_e^a = y$

$x^2 = \frac{y}{b\log_e^a}$

$\Rightarrow x = \frac{\sqrt{y}}{\sqrt{b\log_e^a}}$

$dx = \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{b\log_e^a}} dy$

$x = 0 \quad y = 0$
$x \to \infty \quad y \to \infty$

$= \int_0^\infty e^{-y} \frac{1}{2\sqrt{y}} \frac{dy}{\sqrt{b\log_e^a}}$

$= \frac{1}{2\sqrt{b\log_e^a}} \int_0^\infty e^{-y} \cdot y^{-1/2} dy$

$= \frac{1}{2\sqrt{b\log_e^a}} \cdot \int_0^\infty e^{-x} x^{\frac{1}{2}-1} dx$

$= \frac{1}{2\sqrt{b\log_e^a}} \Gamma\left(\frac{1}{2}\right) \qquad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx\right]$

$= \frac{\sqrt{\pi}}{2\sqrt{b\log_e^a}}$

$$\boxed{\therefore \int_0^\infty a^{-bx^2} dx = \frac{\sqrt{\pi}}{2\sqrt{b\log_e^a}}}$$

Q Express the integral $\int_0^\infty \frac{x^c}{c^x}\,dx$ in terms of Γ function.

<u>Sol</u>

$$c = e^{\log_e^c}$$

$$c^x = \left[e^{\log_e^c}\right]^x$$

$$c^x = e^{x\log_e^c}$$

$$\Rightarrow \int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty \frac{x^c}{e^{x\log_e^c}}\cdot dx$$

$$= \int_0^\infty x^c e^{-x\log_e^c}\,dx$$

Put $x\log_e^c = y$

$$dx = \frac{dy}{\log_e^c}$$

when $x=0$ $y=0$

$x\to\infty$ $y\to\infty$

$$= \int_0^\infty \left[\frac{y}{\log_e^c}\right]^c \cdot e^{-y}\,\frac{dy}{\log_e^c}$$

$$= \frac{1}{\left[\log_e^c\right]^{c+1}} \int_0^\infty y^c e^{-y}\,dy$$

WKT $\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx}$

$$= \frac{1}{\left[\log_e^c\right]^{c+1}} \int_0^\infty x^{(c+1)-1} e^{-x}\,dx$$

$$= \frac{1}{(\log_e^c)^{c+1}}\,\Gamma(c+1)$$

$$\boxed{\therefore \int_0^\infty \frac{x^c}{c^x}\,dx = \frac{1}{(\log_e^c)^{c+1}}\,\Gamma(c+1)}$$

12) evaluate the following integrals

i) $\int_0^1 x^3 \sqrt{1-x}\, dx$

ii) $\int_0^\infty \cdot \frac{x^4(1+x^5)}{(1+x)^{15}} dx$

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iii) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx$

soln: iii) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx$

$= \int_0^\infty \left(\frac{x^8 - x^{14}}{(1+x)^{24}} \right) dx$

$= \int_0^\infty \frac{x^8}{(1+x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}} dx$

$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}$

$= B(9,15) - B(15,9) \quad \left[\because \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n) \right]$

$= \frac{\gamma(9) \cdot \gamma(15)}{\gamma(9+15)} - \frac{\gamma(15)\gamma(9)}{\gamma(15+9)}$

$= 0$

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ii) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$

$= \int_0^\infty \left(\frac{x^4+x^9}{(1+x)^{15}}\right)dx$

$= \int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx$

$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$

$= B(5,10) + B(10,5)$

$= 2B(5,10)$ $[\because B(m,n) = B(n,m)]$

$= 2 \cdot \frac{\gamma(5)\gamma(10)}{\gamma(5+10)}$

$= \cancel{2} \cdot \frac{4!\,\cancel{9!}}{\cancel{14}_{7} \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot \cancel{9!}} = \frac{\cancel{4} \cdot \cancel{3} \cdot \cancel{2}}{7 \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10}_{5}}$.

$= \frac{1}{5005}$

i) $\int_0^1 x^3\sqrt{1-x}\,dx$

$\int_0^1 x^3(1-x)^{1/2}\,dx$

$= \int_0^1 x^{4-1}(1-x)^{3/2-1}\,dx$

$= B(4, \frac{3}{2})$ $[\because B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx]$.

$= \frac{\gamma(4)\gamma(3/2)}{\gamma(4+3/2)}$

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$$= \frac{3!\, \gamma\left(\frac{3}{2}\right)}{\gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\cancel{6}^{2} \cdot \gamma\left(\frac{3}{2}\right)}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{\cancel{3}}{2}\cdot\frac{1}{2}\cdot\gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\cancel{2} \cdot \cancel{\frac{1}{2}\, \gamma\left(\frac{1}{2}\right)}}{\frac{315}{16}\, \frac{1}{2}\cdot\cancel{\gamma\left(\frac{1}{2}\right)}} = \frac{2}{1} \times \frac{16}{315}$$

$$= \frac{32}{315}$$

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13) Evaluate the following integrals.

(i) $\int_0^\infty e^{-x^2} x^{7/2}\, dx$ $\quad x^2 = y$

(ii) $\int_0^\infty \frac{x\,dx}{(1+x^6)}$ $\quad x^6 = y$

14) Prove that $\int_0^\infty \sqrt{x}\, e^{-x^2} dx = 2\int_0^\infty x^2 e^{-x^4}$ using Beta and Gamma functions.

Sol^n: Let $I_1 = \int_0^\infty \sqrt{x}\, e^{-x^2} dx$

Let $x^2 = y \Rightarrow x = y^{1/2}$

$2x\,dx = dy \Rightarrow dx = \frac{dy}{2x}$

$dx = \frac{dy}{2y^{1/2}}$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$I_1 = \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \int_0^\infty \sqrt{y^{1/2}}\, e^{-y} \left(\frac{dy}{2y^{1/2}}\right.$

$$= \frac{1}{2}\int_0^\infty y^{1/4}\,\frac{e^{-y}}{y^{1/2}}\,dy$$

$$= \frac{1}{2}\int_0^\infty y^{-1/4}\,e^{-y}\,dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\,y^{-1/4}\,dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\,y^{\left(\frac{3}{4}-1\right)}\,dy$$

$$I_1 = \frac{1}{2}\Gamma\left(\frac{3}{4}\right) \text{ --- } (1) \qquad \left[\because \Gamma(n) = \int_0^\infty e^{-x}x^{(n-1)}dx\right]$$

Let $I_2 = 2\int_0^\infty x^2 e^{-x^4}\,dx$

Let $x^4 = y \Rightarrow x = y^{1/4}$

$4x^3 dx = dy \Rightarrow dx = \dfrac{dy}{4x^3}$

$dx = \dfrac{dy}{4y^{3/4}}$ $\qquad x=0 \Rightarrow y=0$

$\qquad x=\infty \Rightarrow y=\infty$

$$I_2 = 2\int_0^\infty x^2 e^{-x^4}dx = 2\int_0^\infty (y^{1/4})^2\, e^{-y}\cdot\frac{dy}{4y^{3/4}}$$

$$= \frac{2}{4}\int_0^\infty y^{2/4-\frac{3}{4}}\,e^{-y}\,dy$$

$$= \frac{1}{2}\int_0^\infty y^{-1/4}\,e^{-y}\,dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\,y^{\left(\frac{3}{4}-1\right)}\,dy$$

$$I_2 = \frac{1}{2}\Gamma\left(\frac{3}{4}\right) \text{ --- } (2)$$

from eqⁿ ① and ②

① = ②

$$\therefore \int_0^\infty \sqrt{x}\, e^{-x^2} dx = 2\int_0^\infty x^2 e^{-x^4} dx$$

13/01/19:

1) $\int_0^\infty e^{-x^2} x^{7/2} dx$

put $x^2 = y \Rightarrow x = y^{1/2}$

$2x\,dx = dy \Rightarrow dx = \frac{dy}{2x} = \frac{dy}{2y^{1/2}}$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\int_0^\infty e^{-x^2} x^{7/2} dx = \int_0^\infty e^{-y} (y^{1/2})^{7/2} \frac{dy}{2y^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty e^{-y} y^{\frac{7}{4}-\frac{1}{2}} dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y} y^{\frac{5}{4}} dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y} y^{\left(\frac{9}{4}-1\right)} dy$$

$$\int_0^\infty e^{-x^2} x^{7/2} dx = \frac{1}{2}\Gamma\left(\frac{9}{4}\right) \qquad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{(n-1)} dx\right]$$

$$= \frac{1}{2}\left(\frac{5}{4}\right)\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{5}{32}\Gamma\left(\frac{1}{4}\right)$$

ii) $\int_0^\infty \frac{x\,dx}{(1+x^6)}$

put $x^6 = y \Rightarrow x = y^{1/6}$

$6x^5 dx = dy \Rightarrow dx = \frac{dy}{6x^5} = \frac{dy}{6y^{5/6}}$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\int_0^\infty \frac{x\,dx}{(1+x^6)} = \int_0^\infty \frac{y^{1/6}}{(1+y)} \frac{dy}{6y^{5/6}}$$

$$= \frac{1}{6}\int_0^\infty y^{1/6-5/6} (1+y)^{-1} dy$$

$$= \frac{1}{6}\int_0^\infty y^{-2/3} (1+y)^{-1} dy$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{-\frac{2}{3}}}{(1+y)} dy \quad \left[\because B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx\right]$$

$n-1 = -\frac{2}{3}$ $\qquad m+n = 1$

$n = -\frac{2}{3}+1$ $\qquad m+\frac{1}{3} = 1$

$n = \frac{1}{3}$ $\qquad m = 1-\frac{1}{3}$

$m = \frac{2}{3}$

$$= \frac{1}{6}\int_0^\infty \frac{y^{\frac{1}{3}-1}}{(1+y)^{\frac{2}{3}+\frac{1}{3}}}$$

$$= \frac{1}{6} B\left(\frac{2}{3}, \frac{1}{3}\right)$$

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15) prove that $B(n,n) = \frac{\sqrt{\pi}\,\sqrt{(n)}}{2^{2n-1}\,\sqrt{(n+\frac{1}{2})}}$

sol^n: By the definition of beta-function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Now $B(n,n) = \int_0^1 x^{n-1}(1-x)^{n-1}\,dx$ — ①

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$① \Rightarrow B(n,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{n-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$$= \frac{2}{2^{2n-1}}\int_0^{\frac{\pi}{2}} 2^{2n-1}\sin^{2n-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$$= \frac{2}{2^{2n-1}}\int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^{2n-1}\,d\theta$$

$$= \frac{2}{2^{2n-1}}\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)\,d\theta$$

put $2\theta = \phi$

$\theta = \frac{\phi}{2}$

$d\theta = \frac{1}{2}d\phi$

and $\theta = 0 \Rightarrow \phi = 0$

$\theta = \frac{\pi}{2} \Rightarrow \phi = \pi$

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$$B(n,n)=\frac{2}{2^{2n-1}}\int_{\phi=0}^{\pi}\sin^{2n-1}\phi\,\frac{d\phi}{2}$$

$$=\frac{2}{2^{2n-1}}\cdot\frac{1}{\cancel{2}}(\cancel{2})\int_{0}^{\frac{\pi}{2}}\sin^{2n-1}\phi\,d\phi \quad \left[\because \int_{0}^{\pi}f(x)dx=2\int_{0}^{\frac{\pi}{2}}f(x)dx\right]$$

$$=\frac{2}{2^{2n-1}}\int_{0}^{\frac{\pi}{2}}\sin^{2n-1}\phi\,d\phi$$

$$=\frac{2}{2^{2n-1}}\int_{0}^{\frac{\pi}{2}}\sin^{2n-1}\theta\,d\theta \quad \left[\because \int_{a}^{b}f(t)dt=\int_{a}^{b}f(x)dx\right]$$

By using $\quad B(m,n)=2\int_{0}^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

$m=n,\ n=\frac{1}{2}$

$B(n,\frac{1}{2}) \Rightarrow \frac{\gamma(n)\gamma(\frac{1}{2})}{\gamma(n+\frac{1}{2})}$

$$B(n,n)=\frac{\cancel{2}}{2^{2n-1}}\cdot\frac{\sqrt{\pi}}{\cancel{2}}\cdot\frac{\gamma(n)}{\gamma(n+\frac{1}{2})} \quad \left[\because \int_{0}^{\frac{\pi}{2}}\sin^{2n-1}\theta\,d\theta=\frac{\sqrt{\pi}}{2}\frac{\gamma(n)}{\gamma(n+\frac{1}{2})}\right]$$

$$\therefore B(n,n)=\frac{\sqrt{\pi}\,\gamma(n)}{2^{2n-1}\gamma(n+\frac{1}{2})}$$

Legendre's Duplication formula:-

problems

1) Prove that $2^{2n-1}\gamma(n)\gamma(n+\frac{1}{2})=\gamma(2n)\sqrt{\pi}$ (or)

show that $\gamma(\frac{1}{2})\gamma(2n)=2^{2n-1}\gamma(n)\gamma(n+\frac{1}{2})$ (or)

show that $\gamma(m)\gamma(m+\frac{1}{2})=\frac{\sqrt{\pi}}{2^{2m-1}}\gamma(2m)$

Sol^n: By the definition of beta function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\Rightarrow \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \quad \text{—①} \quad [\text{using } B(m,n) = B(n,m)]$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{n-1}(1-\sin^2\theta)^{m-1}\, 2\sin\theta\cos\theta\,d\theta = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^{(2n-2)}\theta\,\cos^{2m-2}\theta\,\sin\theta\cos\theta\,d\theta = \frac{\Gamma(n)\cdot\Gamma(m)}{\Gamma(n+m)}$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,\cos^{2m-1}\theta\,d\theta = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \quad \text{—② say}$$

Putting $m = \frac{1}{2}$ in eq^n ② we get

$$② \Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,d\theta = \frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} \quad \text{—③}$$

Putting $m = n$ in eq^n ② we get

$$② \Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(n)\Gamma(n)}{\Gamma(n+n)}$$

$$\Rightarrow \frac{2}{2^{2n-1}}\int_0^{\frac{\pi}{2}} 2^{2n-1}\sin^{2n-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{(\gamma(n))^2}{\gamma(2n)}$$

$$\Rightarrow \frac{2}{2^{2n-1}}\int_0^{\frac{\pi}{2}} \sin^{2n-1}(2\theta)d\theta = \frac{(\gamma(n))^2}{\gamma(2n)}$$

put $2\theta = \phi \Rightarrow \theta = \frac{\phi}{2}$

$d\theta = \frac{d\phi}{2}$

and $\theta = 0 \Rightarrow \phi = 0$

$\theta = \frac{\pi}{2} \Rightarrow \phi = \pi$

$$\Rightarrow \frac{2}{2^{2n-1}}\int_0^{\pi} \sin^{(2n-1)}\phi \frac{d\phi}{2} = \frac{(\gamma(n))^2}{\gamma(2n)}$$

$$\Rightarrow \frac{2}{2^{2n-1}}\left(\frac{1}{\cancel{2}}\right)(\cancel{2})\int_0^{\frac{\pi}{2}} \sin^{(2n-1)}\phi\, d\phi = \frac{(\gamma(n))^2}{\gamma(2n)}$$ $\left[\because f(\pi - x) = f(x) \text{ so } \int_0^{\pi} f(x)dx = 2\int_0^{\frac{\pi}{2}} f(x)dx\right]$

$$\Rightarrow \frac{2}{2^{2n-1}}\int_0^{\frac{\pi}{2}} \sin^{(2n-1)}\theta\, d\theta = \frac{(\gamma(n))^2}{\gamma(2n)}$$

$$\Rightarrow \frac{\cancel{2}}{2^{2n-1}} \cdot \frac{\sqrt{\pi}}{\cancel{2}}\frac{\cancel{\gamma(n)}}{\gamma(n+\frac{1}{2})} = \frac{(\gamma(n))^{\cancel{2}}}{\gamma(2n)}$$ $[\because \text{eq}^n \text{ 3}]$

$$\Rightarrow 2^{2n-1}\gamma(n)\gamma\left(n+\frac{1}{2}\right) = \gamma(2n)\sqrt{\pi}$$

$$\therefore 2^{2n-1}\gamma(n)\gamma\left(n+\frac{1}{2}\right) = \gamma(2n)\sqrt{\pi}$$

But we have $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

so put $\sqrt{\pi} = \gamma\left(\frac{1}{2}\right)$

$$\therefore 2^{2n-1}\gamma(n)\gamma\left(n+\frac{1}{2}\right) = \gamma(2n)\gamma\left(\frac{1}{2}\right)$$

$\therefore \Gamma(\frac{1}{2})\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})$

2) prove that $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Sol^n: By other form of beta function

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

By the relation b/w $B(m,n)$ and Γ

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{— ① say}$$

Here put $m+n=1 \Rightarrow m = 1-n$

$$① \Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx = \frac{\Gamma(n)\cdot\Gamma(1-n)}{\Gamma(1)}$$

$$\int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx = \Gamma(n)\Gamma(1-n) \quad \text{— ②} \quad [\because \Gamma(1)=1]$$

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n}\,\mathrm{cosec}\,\frac{(2m+1)\pi}{2n}$$ whereas $m>0, n>0$ & $m>n$.

Put $x^{2n} = t$ and $\dfrac{2m+1}{2n} = s$

$\Rightarrow 2n x^{2n-1}\,dx = dt$

$x^{2n-1}\,dx = \dfrac{1}{2n}\,dt$

$\Rightarrow dx = \dfrac{1}{2n\cdot(x^{2n})\,x^{-1}}\,dt$

$$= \frac{1}{2nx^{2n}} x\,dt$$

$$dx = \frac{t^{1/2n}\,dt}{2n \cdot t}$$

and $x=0 \Rightarrow t=0$

$x=\infty \Rightarrow t=\infty$

$$\therefore \int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n} \operatorname{cosec} \frac{(2m+1)\pi}{2n}$$

$$\Rightarrow \int_0^\infty \frac{t^{\frac{2m}{2n}} \cdot t^{1/2n}\,dt}{(1+t)\,2n \cdot t} = \frac{\pi}{2n} \operatorname{cosec}(s\pi)$$

$$\Rightarrow \int_0^\infty \frac{t^{\left(\frac{2m+1}{2n}\right)}\,t^{-1}}{2n(1+t)}\,dt = \frac{\pi}{2n} \operatorname{cosec}(s\pi)$$

$$\Rightarrow \int_0^\infty \frac{t^{\left(\frac{2m+1}{2n}-1\right)}}{\cancel{2n}(1+t)}\,dt = \frac{\pi}{2n} \cdot \frac{1}{\sin(s\pi)}$$

$$\Rightarrow \int_0^\infty \frac{t^{s-1}}{(1+t)}\,dt = \frac{\pi}{\sin(s\pi)}$$

$$\Rightarrow \int_0^\infty \frac{x^{s-1}}{(1+x)}\,dx = \frac{\pi}{\sin(s\pi)}$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx = \frac{\pi}{\sin(n\pi)}$$

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)} \quad [\because \text{By } 2]$$

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3) show that $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4}$

soln: LHS $= \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

put $x^2 = \sin\theta \Rightarrow x = \sin^{1/2}\theta$

$2x\,dx = \cos\theta\, d\theta$

$dx = \frac{\cos\theta\, d\theta}{2x} = \frac{\cos\theta\, d\theta}{2\sin^{1/2}\theta}$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin\theta \cos\theta \sin^{-1/2} d\theta}{2\sqrt{1-\sin^2\theta}} \times \int_0^{\frac{\pi}{2}} \frac{\cos\theta \sin^{-1/2}\theta\, d\theta}{2\sqrt{1-\sin^2\theta}}$$

$$\Rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^{1/2}\theta \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cancel{\cos\theta} \sin^{-1/2}\theta\, d\theta}{\cancel{\cos\theta}}$$

$$\Rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta\, d\theta \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta\, d\theta \quad \text{--- (1)}$$

By the Beta function

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^0\theta\, d\theta \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^0\theta\, d\theta$$

$\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta \cos^0\theta\, d\theta \Rightarrow 2m-1 = \frac{1}{2}, \quad 2n-1 = 0$

$2m = \frac{3}{2}, \quad 2n = 1$

$m = \frac{3}{4}, \quad n = \frac{1}{2}$

$\int_0^{\frac{\pi}{2}} \sin^{1/2} \cos^0\theta\, d\theta = \frac{1}{2} B(m,n) = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right)$

$$\Rightarrow \cancel{\tfrac{1}{2}}\int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{0}\theta\, d\theta = \frac{1}{2}B(m,n)$$

$$\Rightarrow 2m-1 = -\frac{1}{2} \qquad 2n-1=0$$

$$2m = -\frac{1}{2}+1 \qquad 2n=1$$

$$2m = \frac{1}{2} \qquad n=\frac{1}{2}$$

$$m=\frac{1}{4},$$

$$\cancel{\tfrac{1}{2}}\int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta\cos^{0}\theta\, d\theta = \frac{1}{2}B\left(\frac{1}{4},\frac{1}{2}\right)$$

sub in eq^n 1

$$1 \Rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta\, d\theta \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}B\left(\frac{3}{4},\frac{1}{2}\right) \times \frac{1}{2}\cdot\frac{1}{2}B\left(\frac{1}{4},\frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{16}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$\Rightarrow \frac{1}{16}\,\frac{\cancel{\Gamma\left(\frac{3}{4}\right)}\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)} \times \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\cancel{\Gamma\left(\frac{3}{4}\right)}}$$

$$\Rightarrow \frac{1}{16}\,\frac{\cancel{\Gamma\left(\frac{1}{4}\right)}\pi}{\left(\frac{1}{4}\right)\cancel{\Gamma\left(\frac{1}{4}\right)}}$$

$$= \frac{4\pi}{16}$$

$$= \frac{\pi}{4} = \text{RHS}$$

4) Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}dx$ using β-γ functions.

Sol^n: Given $4\int_0^\infty \frac{x^2}{1+x^4}dx$

put $x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\,d\theta \Rightarrow dx = \frac{\sec^2\theta\,d\theta}{2\tan^{1/2}\theta}$

$x=0 \Rightarrow \theta=0$

$x=\infty \Rightarrow \theta=\frac{\pi}{2}$

$$\Rightarrow \cancel{4}_2\int_0^{\frac{\pi}{2}} \frac{\tan\theta}{1+\tan^2\theta}\cdot\frac{\sec^2\theta\,d\theta}{\cancel{2}\tan^{1/2}\theta}$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \frac{\tan^{1/2}\theta}{\cancel{\sec^2}\theta}\cdot\cancel{\sec^2}\theta\,d\theta$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \tan^{1/2}\theta\,d\theta$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

By the Beta function $2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = B(m,n)$

$2m-1=\frac{1}{2}$, $2n-1=-\frac{1}{2}$

$2m=\frac{1}{2}+1$, $2n=-\frac{1}{2}+1$

$2m=\frac{3}{2}$ $2n=\frac{1}{2}$

$m=\frac{3}{4}$, $n=\frac{1}{4}$

$\Rightarrow B\left(\frac{3}{4},\frac{1}{4}\right)$

$$\Rightarrow \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$\Rightarrow \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$\Rightarrow \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

By $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\Rightarrow \frac{\pi}{\sin\left(\frac{1}{4}\right)\pi} = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$\Rightarrow \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \sqrt{2}\pi$$

① Q Prove that $\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi}{n} \cancel{\operatorname{cosec}} \csc\left(\frac{\pi}{n}\right)$

Sol:

Put $x^n = y \Rightarrow x = y^{1/n}$

$dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$

when $x=0, y=0$

$x=1, y=1$

$\int_0^1 \frac{1}{n} y^{\frac{1}{n}-1} dy/(1-y)^{1/n}$

$= \frac{1}{n}\int_0^1 (1-y)^{-1/n} y^{\frac{1}{n}-1} dy$

$= \frac{1}{n}\int_0^1 y^{\frac{1}{n}-1} (1-y)^{\frac{n-1}{n}-1} dy$

$= \frac{1}{n}\int_0^1 x^{\frac{1}{n}-1} (1-x)^{\frac{n-1}{n}-1} dx$

WKT $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

$= \frac{1}{n} B\left(\frac{1}{n}, \frac{n-1}{n}\right)$

$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$= \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{n-1}{n}\right)}$

$= \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma(1)}$

WKT $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$= \frac{1}{n} \frac{\pi}{\sin\frac{\pi}{n}}$

$\therefore \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi}{n} \csc\left(\frac{\pi}{n}\right)$

Q(2). when n is a +ve integer Prove that $2^n\Gamma(n+\frac{1}{2})=1\cdot3\cdot5---(2n-1)\sqrt{\pi}$

Ans: WKT $\Gamma(n+\frac{1}{2})=\Gamma(n-\frac{1}{2}+1)$

$$=(n-\tfrac{1}{2})\Gamma(n-\tfrac{1}{2})$$

$$=(n-\tfrac{1}{2})\Gamma(n-\tfrac{3}{2}+1)$$

$$=(n-\tfrac{1}{2})(n-\tfrac{3}{2})\Gamma(n-\tfrac{5}{2}+1)$$

$$=(n-\tfrac{1}{2})(n-\tfrac{3}{2})\Gamma(n-\tfrac{5}{2})(n-\tfrac{5}{2})$$

$$=\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\Gamma\left(\frac{2n-5}{2}\right)$$

$$=\frac{(2n-1)(2n-3)(2n-5)----5\cdot3\cdot1}{2\cdot2\cdot------2\cdot2---}\Gamma(\tfrac{1}{2})$$

$$\Gamma(n+\tfrac{1}{2})=\frac{(2n-1)(2n-3)(2n-5)---5\cdot3\cdot1}{2^n}\sqrt{\pi}$$

$$2^n\Gamma(n+\tfrac{1}{2})=1\cdot3\cdot5---(2n-1)\sqrt{\pi}$$

Q(3) Show that $\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx=\frac{1\cdot3\cdot5---(n-1)}{8\cdot4\cdot6---n}\frac{\pi}{2}$ where n is an even integer.

Sol $\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx=\int_0^1 x^n(1-x^2)^{-1/2}dx$

Put $x^2=y$

$x=y^{1/2}$

D.w.r.t x

$dx=\frac{1}{2}y^{-1/2}dy$

when $x=0, y=0$ $x\to1; y\to1$

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx=\int_0^1 y^{n/2}(1-y)^{-1/2}\tfrac{1}{2}y^{-1/2}dy$$

$$=\frac{1}{2}\int_0^1 y^{\frac{n-1}{2}}(1-y)^{1/2-1}dy$$

$$=\frac{1}{2}\int_0^1 x^{\frac{n+1}{2}-1}(1-x)^{1/2-1}dx$$

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,dx = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

WKT $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,dx = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

Since n is even put $n=2r$

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{2r+1}{2}\right)}{\Gamma\left(\frac{2r+2}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma(r+1)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(r+1/2)}{r!}$$

$$= \frac{\sqrt{\pi}}{2r!} \Gamma\left(r-\frac{1}{2}+1\right)$$

$$= \frac{\sqrt{\pi}}{2r!} \left(r-\frac{1}{2}\right)\Gamma\left(r-\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2r!} \left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\Gamma\left(r-\frac{3}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2r!} \left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\left(r-\frac{5}{2}\right)\Gamma\left(r-\frac{5}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2r!} \left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\left(r-\frac{5}{2}\right) \cdots \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2r!} \frac{(2r-1)}{2}\frac{(2r-3)}{2}\frac{(2r-5)}{2} \cdots \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$= \frac{\pi}{8} \frac{(2r-1)(2r-3)(2r-5)\cdots 3\cdot 1}{r(r-1)(r-2)\cdots 3\cdot 2\cdot 1 \cdots 2^n \cdots}$$

$$= \frac{\pi}{2} \frac{1\cdot 3\cdot 5 \cdots (2r-5)(2r-3)(2r-1)}{2r(2r-2)(2r-4)\cdots 6\cdot 4\cdot 2}$$

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,dx = \frac{1\cdot 3\cdot 5 \cdots (n-5)(n-3)(n-1)}{2\cdot 4\cdot 6\cdot 8 \cdots (n-4)(n-2)n}\,\frac{\pi}{2}$$

Q Show that $\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx = \frac{2\cdot4\cdot6\cdots(n-1)}{1\cdot3\cdot5\cdots n}$ where n is an odd integer
(7)

sol:

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx = \int_0^1 x^n(1-x^2)^{-1/2}dx$$

put $x^2=y \Rightarrow x=y^{1/2} \Rightarrow dx=\frac{1}{2}y^{-1/2}dy$

$x=0 ; y=0$

$x=1, y=1$

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx = \int_0^1 y^{n/2}(1-y)^{-1/2}\frac{1}{2}y^{-1/2}dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{n-1}{2}}(1-y)^{-1/2}dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{n+1}{2}-1}(1-y)^{1/2-1}dy$$

$$= \frac{1}{2}\int_0^1 x^{\frac{n+1}{2}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$\int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx = \frac{1}{2}B\left(\frac{n+1}{2},\frac{1}{2}\right)$$

WKT $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1+1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

n is odd $n=2r+1$

$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{2r+2}{2}\right)}{\Gamma\left(\frac{2r+3}{2}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(r+1)}{\Gamma(r+\frac{3}{2})}$$

$$= \frac{r!}{\Gamma\left(r+\frac{1}{2}+1\right)}\cdot\frac{\sqrt{\pi}}{2} = \frac{r(r-1)(r-2)\cdots3\cdot2\cdot1}{\Gamma(r+1/2)(r+1/2)}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{r(r-1)(r-2)\cdots3\cdot2\cdot1}{(r+1/2)(r-1/2)\Gamma(r-1/2)}\cdot\frac{\sqrt{\pi}}{2}$$

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$$= \frac{r(r-1)(r-2)\cdots 3\cdot 2\cdot 1}{(r+1/2)(r-1/2)(r-3/2)\cdots \frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{r(r-1)(r-2)\cdots 3\cdot 2\cdot 1}{\left(\frac{2r+1}{2}\right)\left(\frac{2r-1}{2}\right)\left(\frac{2r-3}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \frac{2r(2r-2)(2r-4)\cdots 6\cdot 4\cdot 2}{(2r+1)(2r-1)(2r-3)\cdots 3\cdot 1}$$

$$= \frac{(2(r+1)-1)((2(r+1)-3)((2r+1)-5)\cdots 6\cdot 4\cdot 2}{(2r+1)((2r+1)-2)((2r+1)-4)\cdots 3\cdot 1} \qquad [\because n = 2r+1]$$

$$= \frac{(n-1)(n-3)(n-5)(n-7)\cdots 6\cdot 4\cdot 2}{n(n-2)(n-4)(n-6)\cdots 3\cdot 1.}$$

$$= \frac{2\cdot 4\cdot 6\cdots(n-5)(n-3)(n-1)}{1\cdot 3\cdot 5\cdots(n-4)(n-2)n}$$

$$\therefore \int_0^1 \frac{x^n}{\sqrt{1-x^2}}dx = \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots(n-4)(n-2)\cdot n}.$$

Q(5) Prove that $\Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\Gamma(\frac{3}{n})\cdots\Gamma(\frac{n-1}{n}) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

Sol^n/

$$K = \Gamma(\tfrac{1}{n})\Gamma(\tfrac{2}{n})\Gamma(\tfrac{3}{n})\cdots\Gamma(\tfrac{n-1}{n}) \quad \text{—①}$$

writing eq① in reverse order we get

$$K^2 = \Gamma(\tfrac{1}{n})\Gamma(1-\tfrac{2}{n})\Gamma(1-\tfrac{3}{n})\cdots\Gamma(1-\tfrac{n-2}{n})\Gamma(1-\tfrac{n-1}{n}) \quad \text{—②}$$

Multiplying ① and ② we get

$$K^2 = \Gamma(\tfrac{1}{n})\Gamma(1-\tfrac{1}{n})\Gamma(\tfrac{2}{n})\Gamma(1-\tfrac{2}{n})\cdots\Gamma(\tfrac{n-1}{n})\Gamma(1-\tfrac{n-1}{n})$$

WKT $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$K^2 = \frac{\pi}{\sin\pi/n}\cdot\frac{\pi}{\sin 2\pi/n}\cdots\frac{\pi}{\sin(\frac{n-1}{n})\pi}$$

$$= \frac{\pi^{n-1}}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{n-1}{n}\right)\pi}$$

$$= \frac{\pi^{n-1}}{\frac{n}{2^{n-1}}} \qquad \left[\because \sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{n-1}{n}\right)\pi = \frac{n}{2^{n-1}}\right]$$

$$k^2 = \frac{(2\pi)^{n-1}}{n}$$

$$k = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$$

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$$

4. MULTIPLE INTEGRALS

Double integrals :-

Let $f(x,y)$ is a continuous function defined in a closed and a bounded region 'R' in XY plane & divide the region 'R' into small elementary rectangles by drawing lines parallel to coordinate axes.

Let the total no. of complete rectangles which lie inside the region 'R' is n.

Let δA_i be the area of ith rectangle and (x_i, y_i) be any point in this rectangle,

Consider the sum $S = \sum_{i=1}^{n} f(x_i, y_i)\,\delta A_i$ where —①

$$\delta A_i = \delta x_i \delta y_i$$

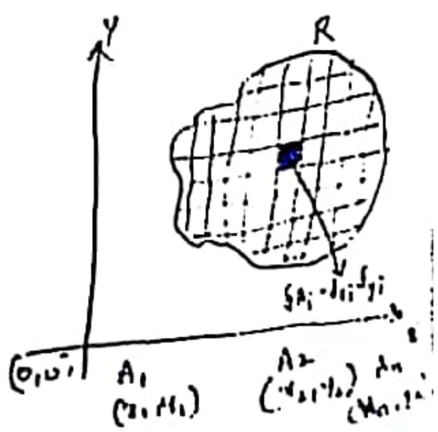
If we increase the no. of elementary rectangles then the area of each rectangle decreases. Hence as $n \to \infty$, $\delta A_i \to 0$.

The limit of the sum given by eq① if it exist is called the double integral of $f(x,y)$ over the region 'R' and is denoted by

$$\iint_R f(x,y)\,dA$$

where $dA = dx\,dy$

Hence $$\iint_R f(x,y)\,dx\,dy = \lim_{\substack{n\to\infty \\ \delta A_i \to 0}} \sum_{i=1}^{n} f(x_i, y_i)\,\delta A_i$$



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$$S = f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \cdots + f(x_n,y_n)\,\delta A_n$$

$$S = \sum_{i=1}^{n} f(x_i, y_i)\,\delta A_i$$

As $n \to \infty$, $\delta A_i \to 0$

Evaluation of double integrals :

Type – ① When all the four limits are constant.

Case – i :

The given double integral is of the form

$$\underbrace{\int_a^b \underbrace{\int_c^d f(x,y)\,dx}_{x}\,dy}_{y} = \int_{y=a}^{y=b}\left[\int_{x=c}^{x=d} f(x,y)\,dx\right]dy.$$

Here first integrate w.r.t 'x' keeping 'y' as constant and then integrate w.r.t 'y'.

Case – ii :

The double integral is of the form

$$\underbrace{\int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{y}\,dx}_{x} = \int_{x=a}^{x=b}\left[\int_{y=c}^{y=d} f(x,y)\,dy\right]dx.$$

Here first integrate w.r.t 'y' keeping 'x' as constant and then integrate w.r.t 'x'.

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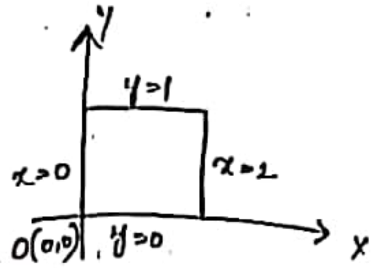
Q) Evaluate $\int_0^1\int_0^2 x^2y^2\,dx\,dy$.

Sol: Let $I = \int_0^1\int_0^2 x^2y^2\,dx\,dy$.



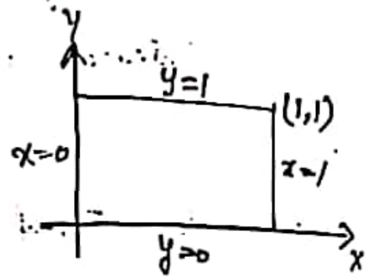
$$= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x^2y^2\,dx\,dy$$

$$= \int_{y=0}^{y=1} y^2\,dy \int_{x=0}^{x=2} x^2\,dx$$

$$= \left[\frac{y^3}{3}\right]_0^1 \left[\frac{x^3}{3}\right]_0^2 = \left(\frac{1}{3}-0\right)\left(\frac{8}{3}-0\right) = \frac{8}{9}.$$

Q) Evaluate $\int_0^1\int_0^1 (x^2+y^2)\,dy\,dx$.

Sol: Let $I = \int_0^1\int_0^1 (x^2+y^2)\,dy\,dx$.



$$= \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1}(x^2+y^2)\,dy\right]dx.$$

$$= \int_{x=0}^{x=1}\left[x^2y + \frac{y^3}{3}\right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=1}\left(x^2+\frac{1}{3}\right)dx = \left[\frac{x^3}{3}+\frac{x}{3}\right]_{x=0}^{x=1} = \left(\frac{1}{3}+\frac{1}{3}\right) - 0 = \frac{2}{3}.$$

Note:

1. When all the four limits are constants we can integrate in any Order.
2. When all the four limits are constants the reason of integration is either a rectangle or square.

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Type - ②

The double integral is of the form

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy.$$

In the given integral the inner integral limits are in terms of x. So these are limits of y.

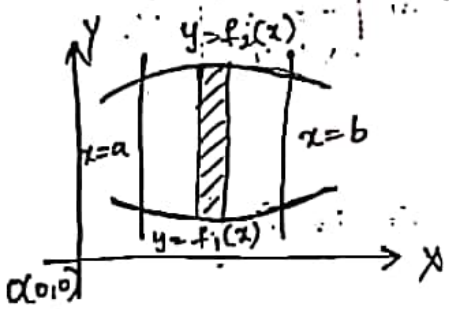
$\therefore$ y limits are $y = f_1(x)$, $y = f_2(x)$.

The outer integral limits are contants $\therefore$ these are limits of x.

$\therefore$ x limits are $x = a$, $x = b$

$$I = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy \right] dx$$

Here first integrate w.r.t 'y' (Keeping x as constant) and then integrate w.r.t 'x'.



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Q) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$.

Sol: Let $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$.

In the given integral the inner integral limits are in terms of x. So these are limits of y.

$\therefore$ y limits are $y=0$, $y=\sqrt{1+x^2}$.

The outer integral limits are constants. these are limits of x.

x limits are $x=0$, $x=1$.

Here first integrate w.r.t y (Keeping x as Constant). and then integrate w.r.t x.

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\, dy \right] dx.$$

$$= \int_{x=0}^{x=1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right) \right]_{y=0}^{y=\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{x=1} \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$= \frac{\pi}{4} \int_{x=0}^{x=1} \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[\sinh^{-1}(x)\right]_{x=0}^{x=1}$$

$$= \frac{\pi}{4} \left[\sinh^{-1}(1) - \sinh^{-1}(0)\right]$$

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Q) Evaluate $\int_0^1 \int_0^x e^{y/x}\, dy\, dx$

Sol: Let $I = \int_0^1 \int_0^x e^{y/x}\, dy\, dx$.

In the given integral the inner integral limits are interms of x. So these are limits of y.

$\therefore$ y limits are $y=0$, $y=x$.

The outer integral limits are constants. these are limits of x

$\therefore$ x limits are $x=0$, $x=1$.

Here first integrate w.r.t y (keeping x as constant). and then integrate w.r.t x.

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=x} e^{y/x}\, dy \right] dx$$

$$= \int_{x=0}^{x=1} \left[x e^{y/x} \right]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=1} x \left[e^0 - e \right] dx = (e^0 - e) \int_{x=0}^{x=1} x\, dx$$

$$= e^0 - e \left[\frac{x^2}{2} \right]_0^1$$

$$= e^0 - e \left[\frac{1}{2} \right] = \frac{1-e}{2}$$

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Type - ③

The double integral is of the form $\int_a^b \int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$.

In the given integral the inner integral limits are in terms of y : so these are limits of x.

$\therefore$ x limits are $x = f_1(y)$, $x = f_2(y)$.

The outer integral limits are constants. these are limits of y.

$\therefore$ y limits are $y=a$, $y=b$.

$$I = \int_a^b \int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy = \int_{y=a}^{y=b} \left[\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx \right] dy.$$

Here first we integrate w.r.t x (keeping y as constant) and then integrate w.r.t y.

Q) Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$.

Sol: Let $I = \int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

In the given integral. the inner integral limits are in terms of y, so these are limits of x.

x limits are $x=0$, $x = \sqrt{\frac{1-y^2}{2}}$.

The outer integral limits are constants. These are limits of y.

y limits are $y=0$, $y=1$.

Here first we integrate w.r.t x (keeping y as constant

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and then integrate w.r.t 'y'.

$$= \int_{y=0}^{y=1} \left[\int_{x=0}^{x=\sqrt{\frac{1-y^2}{2}}} \frac{1}{\sqrt{(1-y^2)-x^2}} dx \right] dy$$

$$= \int_{y=0}^{y=1} \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right) \right]_{x=0}^{x=\sqrt{\frac{1-y^2}{2}}} dx$$

$$= \int_{y=0}^{y=1} \left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0) \right] dy$$

$$= \frac{\pi}{4} \int_{y=0}^{y=1} dy = \frac{\pi}{4} [y]_0^1 = \frac{\pi}{4}.$$

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Type - ②

Evaluation of double integrals in the given region.

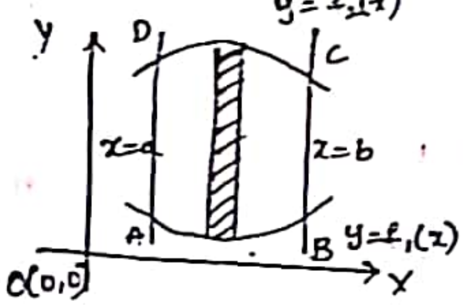
Method - ①

Let the region 'R' i.e ABCD bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$.

In the region ABCD draw a vertical strip PQ along the strip PQ 'y' varies from f_1 to f_2 and x is fixed.

Therefore the double integral is integrated first w.r.t 'y' b/w the limits f_1 and f_2 treating x as constant.

The result of the first integral is integrated w.r.t 'x' b/w the limits a and b.

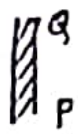


$$\iint f(x,y)\, dx\, dy$$

$$= \int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\, dy \right] dx$$

x limits constant

y limits interms of x

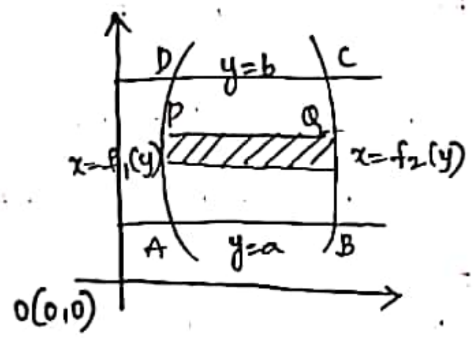


Method - ②.

Let the region 'R' be bounded by the curves $x = f_1(y)$, $x = f_2(y)$ and the lines $y = a$, $y = b$ in the region ABCD draw a horizontal strip PQ along the strip PQ, x varies from f_1 to f_2 and y is fixed.

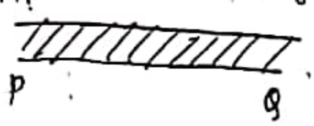
Therefore the double integral is integrated first w.r.t x b/w the limits f_1 and f_2 treating x as constant.

The result of the first integral is integrated w.r.t 'y' b/w the limits a and b.



y limits Constant

x limits interms of y.



$$\iint f(x,y)\,dx\,dy = \int_{y=a}^{y=b}\left[\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\right]dy$$

Q) Evaluate $\iint (x^2+y^2)\,dx\,dy$ in the +ve quadrant for which $x+y \le 1$.

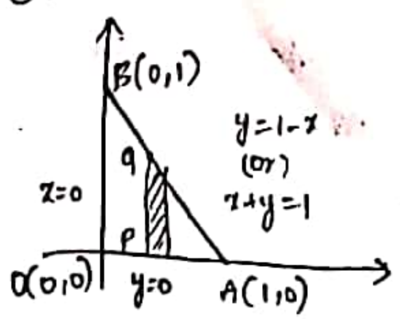
Sol: Let $I = \iint (x^2+y^2)\,dx\,dy$.

let $f(x,y) = x^2+y^2$

The region is in +ve quadrant for which $x+y \le 1$

$\therefore$ the region of integration ΔOAB.

Method - ①.



x limits Constant

y limits interms of x.

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Draw a Vertical strip PQ in the region. we have fix 'x' first in the region x varies from 0 to 1

$\therefore$ x limits are $x=0$, $x=1$.

For each 'x' y varies from a point 'P' on x axis $(y=0)$ to a point Q on the line $x+y=1$.

i.e $y=1-x$

$\therefore$ y limits are $y=0$, $y=1-x$

$$\therefore \quad I = \iint (x^2+y^2)\,dx\,dy = \int_{x=0}^{x=1} \left[\int_{y=0}^{y=1-x} (x^2+y^2)\,dy \right] dx.$$

$$= \int_{x=0}^{x=1} \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=1-x} dx$$

$$= \int_{x=0.}^{x=1} \left[x^2(1-x) + \frac{(1-x)^3}{3} \right] dx$$

$$= \int_{x=0}^{x=1} \left[x^2 - x^3 - \frac{(x-1)^3}{3} \right] dx$$

$$= \left[\frac{x^3}{2} - \frac{x^4}{4} - \frac{(x-1)^4}{12} \right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{4} - 0 \right) - \left(\frac{-1}{12} \right)$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12} = \frac{2}{12} = \frac{1}{6}.$$

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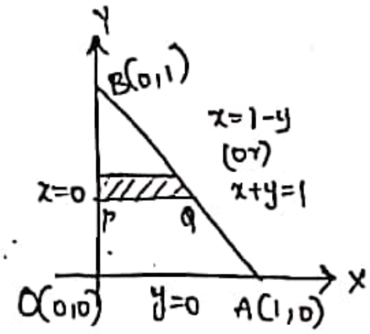
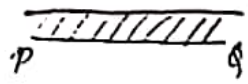
Method - 2





y limits are constant

x limits in terms of y.

Draw a horizontal strip PQ in the region. we have to fix y first in the region y varies from 0 to 1.

$\therefore$ y limits are $y=0, y=1$.

For each y, x varies from a point P on y-axis $(x=0)$ to a point Q on the line $x+y=1$ (or) $x=1-y$.

$\therefore$ x limits are $x=0$, $x=1-y$.

$$I = \iint (x^2+y^2)\,dx\,dy = \int_{y=0}^{y=1}\left[\int_{x=0}^{x=1-y}(x^2+y^2)\,dx\right]dy.$$

$$= \int_{y=0}^{y=1}\left[\frac{2x^3}{3} + y^2x\right]_{x=0}^{x=1-y} dy.$$

$$= \int_{y=0}^{y=1}\left[\frac{x^3}{3} + y^2x\right]_{x=0}^{x=1-y} dy$$

$$= \int_{y=0}^{y=1}\left[\frac{(1-y)^3}{3} + y^2(1-y) - (0+0)\right] dy$$

$$= \int_{y=0}^{y=1}\left(\frac{(1-y)^3}{3} + (-y^3) + y^2\right) dy$$

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$$= \int_{y=0}^{y=1} \frac{(1-y)^3}{3} + y^2 - y^3 \, dy$$

$$= \left[-\frac{(1-y)^4}{3\times 4} + \frac{y^3}{3} - \frac{y^4}{4} \right]_0^1$$

$$= -\frac{(1-1)^4}{3\times 4} + \frac{1}{3} - \frac{1}{4} - \left[-\frac{(1-0)}{3\times 4} + 0 + 0 \right]$$

$$= 0 + \frac{1}{2} - \frac{1}{4} + \frac{1}{12} = \frac{4-3+1}{12} = \frac{2}{12} = \frac{1}{6}.$$

Q) Evaluate $\iint xy\,dx\,dy$ over the +ve quadrant of the circle $x^2+y^2=a^2$

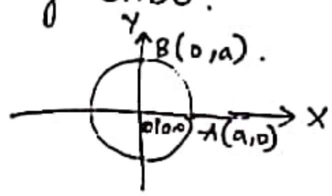
Sol: Let $I = \iint xy\,dx\,dy$

let $f(x,y) = xy$

G.T, $x^2+y^2=a^2$

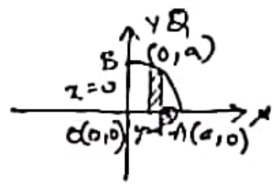
which is a Circle centre at $(0,0)$ and radius 'a' units.

The region of integration is bounded by OABO.



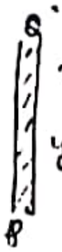
Method -①

Draw a vertical strip PQ in the region,



we have to fix x first.

In the region x varies from 0 to a.



x limits constants

y limits interms of x.

$\therefore$ x limits are $x=0$, $x=a$

For each x, y varies from a point P on x axis ($y=0$) to a point Q on the $\circ^{le}$ $x^2+y^2=a^2$.

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or $y=\sqrt{a^2-x^2}$.

$\therefore$ y limits are $y=0$, $y=\sqrt{a^2-x^2}$.

$\therefore$ $I=\iint xy\,dx\,dy$

$$=\int_{x=0}^{x=a}\left[\int_{y=0}^{y=\sqrt{a^2-x^2}} xy\,dy\right]dx.$$

$$=\int_{x=0}^{x=a} x\,dx\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\int_{x=0}^{x=a} x\left[\left(\sqrt{a^2-x^2}\right)^2-0\right]dx.$$

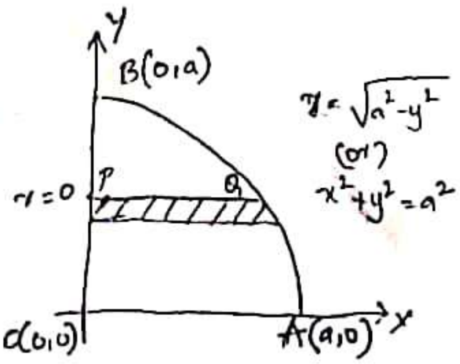
$$=\frac{1}{2}\int_{x=0}^{x=a}\left(a^2x-x^3\right)dx$$

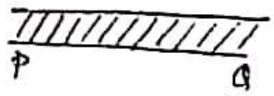
$$=\frac{1}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_{x=0}^{x=a}$$

$$=\frac{1}{2}\left[\left(\frac{a^4}{2}-\frac{a^4}{4}\right)-0\right]$$

$$=\frac{a^4}{8}.$$

Method - ②





y limits constants

x limits interms of y.

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Draw a horizonal strip PQ in the region.
we have to fix y first.
In the region y varies from 0 to a.
$\therefore$ y limits are $y=0$, $y=a$.
For each y x varies from a point P on y axis $(x=0)$
to a point Q on the circle $x^2+y^2=a^2$.
i.e., $x=\sqrt{a^2-y^2}$
$\therefore$ x limits are $x=0$, $x=\sqrt{a^2-y^2}$

$$\therefore I = \iint xy \, dx \, dy$$

$$= \int_{y=0}^{y=a} \left[\int_{x=0}^{x=\sqrt{a^2-y^2}} xy \, dx \right] dy.$$

$$= \int_{y=0}^{y=a} y\,dy \left[\frac{x^2}{2} \right]_{x=0}^{x=\sqrt{a^2-y^2}}$$

$$= \frac{1}{2} \int_{y=0}^{y=a} y \left[\left(\sqrt{a^2-y^2}\right)^2 - 0 \right] dy$$

$$= \frac{1}{2} \int_{y=0}^{y=a} (a^2y - y^3) \, dy$$

$$= \frac{1}{2} \left[a^2\frac{y^2}{2} - \frac{y^4}{4} \right]_{y=0}^{y=a}$$

$$= \frac{1}{2} \left[\left(\frac{a^4}{2} - \frac{a^4}{4} \right) - 0 \right]$$

$$= \frac{a^4}{8}$$

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8) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area bounded b/w the parabola $y=x^2$ and the line $y=x$.

Sol: Let $I = \iint xy(x+y)\,dx\,dy$.

Let $f(x,y) = xy(x+y)$.

The region is bounded by the parabola $y=x^2$ —①.

and the line $y=x$ —②.

The points of intersection of ① & ② is given by,

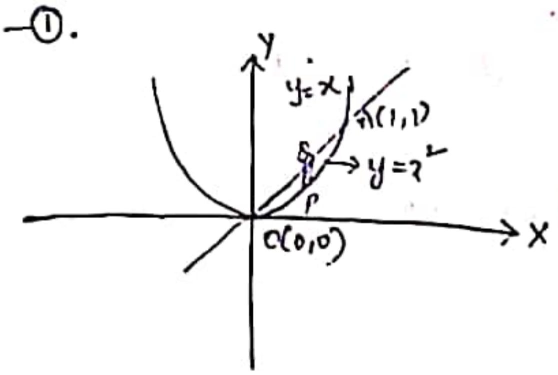
$$y=x^2 \Rightarrow x^2=x$$
$$x(x-1)=0$$
$$x=0,\ x=1.$$

when $x=0$, $y=0$

when $x=1$, $y=1$.

The points of intersection of ① & ② is $O(0,0)$ and $A(1,1)$.

Method —①.



Draw a Vertical Strip PQ in the region.

we have to fix x first.

In the region x varies from 0 to 1.

$\therefore$ x limits are $x=0$, $x=1$

For each x, y varies from a point P on the parabola ($y=x^2$) to a point Q on the line $y=x$.

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$\therefore$ y limits are $y = x^2$, $y = x$.

$\therefore$ $I = \iint xy(x+y)\,dx\,dy$:

$$= \int_{x=0}^{x=1}\left[\int_{y=x^2}^{y=x}(x^2y + xy^2)\,dy\right]dx.$$

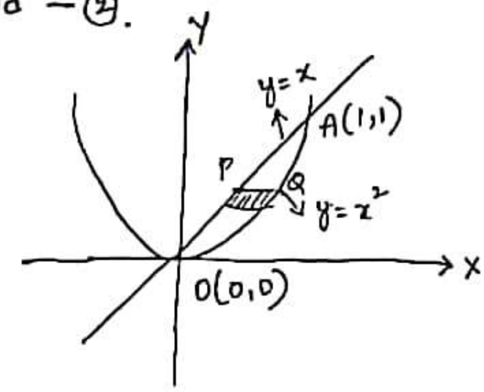
$$= \int_{x=0}^{x=1}\left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_{y=x^2}^{y=x} dx.$$

$$= \int_{x=0}^{x=1}\left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right]dx$$

$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right]_{x=0}^{x=1}$$

$$= \left[\frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}\right] - 0 \quad = \frac{3}{56}$$

Method – 2.



Draw a horizontal PQ in the region. ~~y +~~

we have to fix y first,

In the region y Varies from 0 to 1.

$\therefore$ y limits are $y=0$, $y=1$ for each 'y', x varies from a point P on the parabola ($y = x$) to a point Q on the parabola $y = x^2$ i.e $x = \sqrt{y}$

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$\therefore$ x limits are $x=y$, $x=\sqrt{y}$.

$$I = \iint xy(x+y)\,dx\,dy = \int_{y=0}^{y=1}\left[\int_{x=y}^{x=\sqrt{y}} (x^2y + y^2x)\,dx\right] dy$$

$$= \int_{y=0}^{y=1}\left[y\frac{x^3}{3} + y^2\frac{x^2}{2}\right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_{y=0}^{y=1}\left[\left(\frac{y^2\sqrt{y}}{3} + \frac{y^3}{2}\right) - \left(\frac{y^4}{3} + \frac{y^4}{2}\right)\right] dy$$

$$= \int_{y=0}^{y=1}\left[\frac{y^2\sqrt{y} - y^4}{3} + \frac{y^3 - y^4}{2}\right] dy.$$

$$= \left[\frac{1}{3}\left(\frac{2}{7}y^{7/2} - \frac{y^5}{5}\right) + \frac{1}{2}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\right]_{y=0}^{y=1}$$

$$= \left[\frac{2}{21}y^{7/2} - \frac{y^5}{15} + \frac{y^4}{8} - \frac{y^5}{10}\right]_{y=0}^{y=1}$$

$$= \left(\frac{2}{21} - \frac{1}{15} + \frac{1}{8} - \frac{1}{10}\right) - 0$$

$$= \frac{3}{56}$$

Q) Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabola $y^2 = 4ax$ and $x^2 = 4ay$.

Sol: Let $I = \iint y\,dx\,dy$.

$f(x,y) = y$.

The region is bounded by the parabola $y^2 = 4ax$ — ① and $x^2 = 4ay$ — ②

The points of intersection of ① & ② is,

$y^2 = 4ax \Rightarrow y^4 = 16a^2x^2$

$y^4 = 16a^2(4ay)$

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$y(y^3 - 64a^3) = 0$

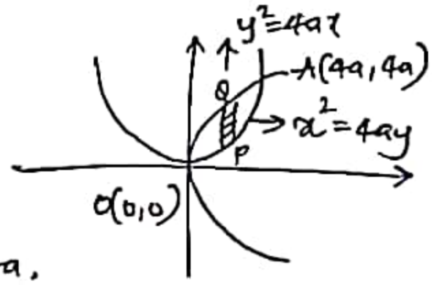
$y = 0, \; y = 4a$

when $y = 0, \; x = 0$

when $y = 4a, \; x = 4a.$

$\therefore$ Point of intersection of ① & ② is $O(0,0)$ & $A(4a, 4a)$.

<u>Method - ①</u> :



Draw a Vertical strip PQ in the region, we have to fix x first in the region x varies from 0 to 4a.

$\therefore$ x limits are $x = 0, \; x = 4a$.

For each x, y varies from a point P on parabola $y = \frac{x^2}{4a}$ to a point Q on parabola $y^2 = 4ax$ i.e $y = 2\sqrt{ax}$

$\therefore \; I = \iint y \, dx \, dy.$

$$= \int_{x=0}^{x=4a} \left[\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} y \, dy \right] dx$$

$$= \int_{x=0}^{x=4a} \left[\frac{y^2}{2} \right]_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx$$

$$= \frac{1}{2} \int_{x=0}^{x=4a} \left[4ax - \frac{x^4}{16a^2} \right]$$

$$= \frac{1}{2} \left[4a \frac{x^2}{2} - \frac{x^5}{80a^2} \right]_{x=0}^{x=4a}$$

$$= \frac{1}{2} \left[\frac{4a \times 16a^2}{2} - \frac{1024 \; a^5}{80 \; a^2} \right]$$

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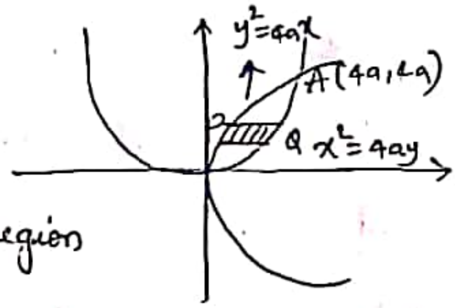
$= \frac{1}{2}\left[32a^3 - \frac{64}{5}a^3\right] = \frac{1}{2}\left[\frac{160a^3 - 64a^3}{5}\right] = \frac{48}{5}a^3.$

Method - 2

Draw a horizontal strip PQ in the region.



we have to fix y first in the region

y varies from 0 to 4a :

$\therefore$ y limits are $y=0$, $y=4a$.

For each y, x varies from a point P on parabola $x = \frac{y^2}{4a}$ to a point Q on parabola $x^2 = 4ay$ i.e $x = 2\sqrt{ay}$

$\therefore$ x limits are $x = \frac{y^2}{4a}$, $x = 2\sqrt{ay}$

$$I = \iint y \, dx \, dy$$

$$= \int_{y=0}^{y=4a} \left[\int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} y \, dx\right] dy$$

$$= \int_{y=0}^{y=4a} y \, [x]_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} dy$$

$$= \int_{y=0}^{y=4a} y\left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$$

$$= \left[2\sqrt{a}\left(\frac{2}{5}y^{5/2}\right) - \frac{1}{4a}\cdot\frac{y^4}{4}\right]_0^{4a}$$

$$= \frac{16}{5}\times 4^{5/2}\, a^3 - 16a^3$$

$$= \frac{16\times 32}{5} a^3 - 16a^3$$

$$= \frac{128}{5} a^3 - 16a^3$$

$$= \frac{48}{5} a^3$$

Q) If R is the Δ^{lar} region with vertices $(0,0)$, $(2,0)$, $(2,3)$
Evaluate $\iint_R x^2y^2\, dx\, dy$.

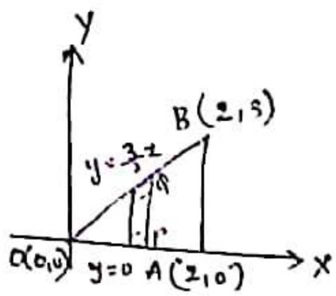
Sol: Let $I = \iint_R x^2y^2\, dx\, dy$.

Let $f(x,y) = x^2y^2$.

G.T, R is the Δ^{lar} region with Vertices $O(0,0)$ $A(2,0)$ $B(2,3)$

Eq. of line OB is $y = \frac{3}{2}x$.

Method - ①.



Draw a Vertical Strip PQ in the region.

We have to fix x first.

In the region x varies from 0 to 2

$\therefore$ x limits are $x=0$, $x=2$.

x limits constant

y limits interms of x.

For each x y varies from a point P on x-axis ($y=0$) to a point Q on the line $y = \frac{3}{2}x$

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$\therefore$ y limits are $y=0$, $y=\frac{3}{2}x$.

$\therefore$ If $I=\iint_R x^2y^2\,dx\,dy$

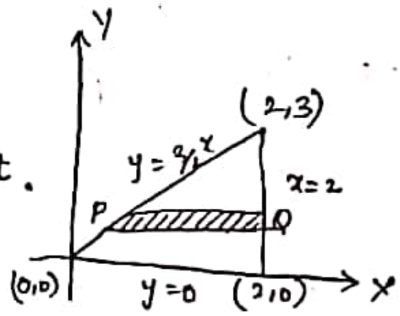
$$= \int_{x=0}^{x=2}\left[\int_{y=0}^{y=\frac{3}{2}x} x^2y^2\,dy\right]dx.$$

$$= \int_{x=0}^{x=2} x^2\left[\frac{y^3}{3}\right]_{y=0}^{y=\frac{3}{2}x} dx$$

$$= \frac{1}{3}\int_{x=0}^{x=2} x^2\left[\left(\frac{3}{2}x\right)^3-0\right]dx$$

$$= \frac{9}{8}\int_{x=0}^{x=2} x^5\,dx = \frac{9}{8}\left[\frac{x^6}{6}\right]_0^2 = \frac{9}{8}\left[\frac{2^6}{6}-0\right] = 12$$

Method - ②.



Draw a horizontal strip PQ in the region we have to fix y first. In the region y varies from 0 to 3.

$\therefore$ y limits are $y=0$, $y=3$.

For each y x varies from at point P on the line $y=\frac{3}{2}x$ i.e $x=\frac{2}{3}y$ to a point Q on the line $x=2$

$\therefore$ x limits are $x=\frac{2}{3}y$, $x=2$

$$\therefore I=\iint_R x^2y^2\,dx\,dy = \int_{y=0}^{y=3}\left[\int_{x=\frac{2}{3}y}^{x=2} x^2y^2\,dx\right]dy$$

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$$= \int_{y=0}^{y=3} y^2 \left[\frac{x^3}{3}\right]_{x=\frac{2}{3}y}^{x=2} dy$$

$$= \int_{y=0}^{y=3} y^2 \left[\frac{8}{3} - \frac{8}{81} y^3\right] dy$$

$$= \int_{y=0}^{y=3} \left(\frac{8}{3} y^2 - \frac{8}{81} y^5\right) dy$$

$$= \left[\frac{8}{3}\frac{y^3}{3} - \frac{8}{81}\frac{y^6}{6}\right]_0^3$$

$$= \frac{8}{3} \times \frac{27}{3} - \frac{8}{81} \times \frac{729}{6}$$

$$= 24 - \frac{5832}{486}$$

$$= 24 - 12$$

$$= 12$$

Q) Evaluate $\iint_R xy \, dx\, dy$ where R is the region bounded by the x axis, line $y = 2x$ on the parabola $x^2 = 4ay$.

Sol: Let $I = \iint_R xy\, dx\, dy$.

Let $f(x,y) = xy$.

The region is bounded by parabola $x^2 = 4ay$ — ① & the line $y = 2x$ — ②.

The points of intersection of ① & ② is given by,

$$x^2 = 4ay \Rightarrow x^2 = 4a(2x)$$

$$x^2 - 8ax = 0$$

$$x(x - 8a) = 0$$

$$x = 0, \; x = 8a.$$

when $x = 0$, $y = 0$

$x = 8a$, $y = 16a$.

The points of intersection of ① & ② is $O(0,0)$ & $A(8a, 16a)$.

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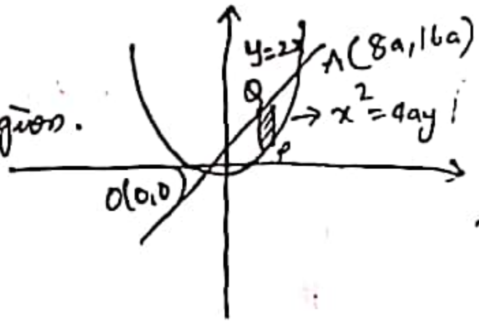
Method - 1



Draw a vertical strip PQ in the region.
we have to fix x first.
On the region x from 0 to 8a

$\therefore$ x limits are $x=0$, $x=8a$

For each x, y varies from point P on the parabola $x^2=4ay$ to a point Q on the line $y=2x$.

$\therefore$ y limits are $y=\frac{x^2}{4a}$, $y=2x$.

$$\therefore \quad I = \iint xy \, dx \, dy$$

$$= \int_{x=0}^{x=8a} \int_{y=\frac{x^2}{4a}}^{y=2x} xy \, dy \, dx$$

$$= \int_{x=0}^{x=8a} \left[\frac{xy^2}{2} \right]_{y=\frac{x^2}{4a}}^{y=2x} dx$$

$$= \frac{1}{2} \int_{x=0}^{x=8a} \left[4x^3 - \frac{x^5}{16a^2} \right] dx$$

$$= \frac{1}{2} \left[4\frac{x^4}{4} - \frac{x^6}{96a^2} \right]_{x=0}^{x=8a}$$

$$= \frac{1}{2} \left[4096a^4 - \frac{262144a^4}{96} \right]$$

$$= \frac{1}{2} \left[4096\,a^4 - \frac{8192\,a^4}{3\cancel{96}} \right]$$

$$= \frac{a^4}{2} \left[\frac{12288 - 8192}{3} \right]$$

$$= \frac{4096\,a^4}{6} = \frac{2048\,a^4}{3}$$

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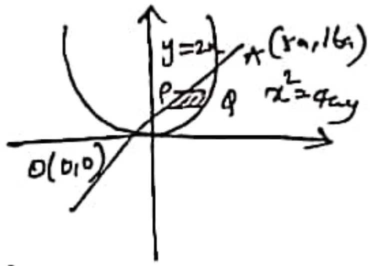
Method - ②



Draw a horizontal strip PQ in the region.

we fix y first.

In the region y varies from 0 to 16a.

$\therefore$ y limits are $y=0$, $y=16a$

For each y, x varies from at point P on the line $y=2x$ i.e $x=\frac{y}{2}$ to the point Q on the line $x=2\sqrt{ay}$.

$\therefore$ x limits are $x=\frac{y}{2}$, $x=2\sqrt{ay}$.

$$\therefore I = \iint_R xy\,dx\,dy$$

$$= \int_{y=0}^{y=16a}\int_{x=\frac{y}{2}}^{x=2\sqrt{ay}} xy\,dx\,dy.$$

$$= \int_{y=0}^{y=16a}\left[\frac{x^2y}{2}\right]_{x=\frac{y}{2}}^{x=2\sqrt{ay}} dy$$

$$= \int_{y=0}^{y=16a}\frac{1}{2}\left[4ay^2-\frac{y^3}{4}\right]$$

$$= \frac{1}{2}\left[\frac{4a}{3}y^3-\frac{y^4}{16}\right]_0^{16a}$$

$$= \frac{1}{2}\left[\frac{4}{3}a\times 4096a^3-\frac{65536}{16}\right]$$

$$= \frac{1}{2}\left[\frac{16384a^4-12288a^4}{3}\right]$$

$$= \frac{4096a^4}{6} = \frac{2048}{3}a^4.$$

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Q) Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by the parabola $y=x^2$ and the line $x+y=2$ and $0 \le x \le 1$.

Sol: Let $I = \iint_R xy\,dx\,dy$

$f(x,y) = xy$

G.T, the parabola $y=x^2$ — ①.

and the line $x+y=2$ — ②.

The points of intersection of ① & ② is given by

$y=x^2 \Rightarrow x^2 = 2-x$

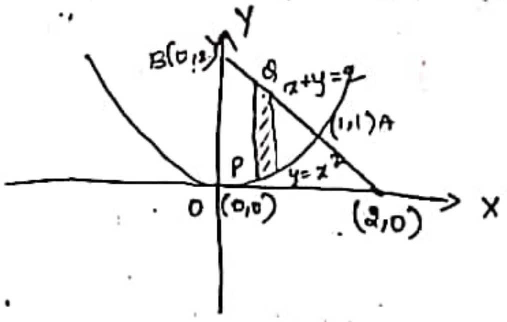
$x = 1, -2.$

when $x=1$, $y=1$

when $x=-2$, $y=4$

∴ The points of intersection of ① & ② is $(1,1), (-2,4)$.

Method -①:



Draw a vertical strip PQ in the region.

we have to fix x first.

In the region 'x' varies from (0,1) x varies from 0 to 1.

x limits are $x=0$ & $x=1$.

For each x, y varies from a point P on the parabola $y=x^2$ to a point Q on the line $x+y=2$.

i.e, $y = 2-x$.

$$\therefore I = \iint_R xy\,dx\,dy = \int_{x=0}^{x=1}\left[\int_{y=x^2}^{y=2-x} xy\,dy\right]dx$$

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$$= \int_{x=0}^{x=1} x\left[\frac{y^2}{2}\right]_{y=x^2}^{y=2-x} dx.$$

$$= \frac{1}{2}\int_{x=0}^{x=1} x\left[(2-x)^2 - x^4\right] dx.$$

$$= \frac{1}{2}\int_{x=0}^{x=1} \left[4x + x^3 - 4x^2 - x^5\right] dx$$

$$= \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^6}{6}\right]_{x=0}^{x=1}$$

$$= \frac{1}{2}\left[\left(2 + \frac{1}{4} - \frac{4}{3} - \frac{1}{6}\right) - 0\right] = \frac{1}{2}\left[\frac{9}{4} - \frac{8-1}{6}\right] = \left(\frac{9}{4} - \frac{7}{6}\right)\frac{1}{2}$$

$$= \left(\frac{9\times 6 - 7\times 4}{24}\right)\frac{1}{2} = \frac{54-28}{24\times 2} = \frac{\cancel{26}^{13}}{24\times\cancel{2}} = \frac{\cancel{13}}{\cancel{24}}$$

$$= \frac{3}{8}.$$

Q) Evaluate $\iint e^{y^2}\, dx\, dy$ over the region bounded by the triangle with vertices $(0,0)$ $(0,1)$ $(2,1)$.

Sol: Let $I = \iint e^{y^2} dx\, dy.$

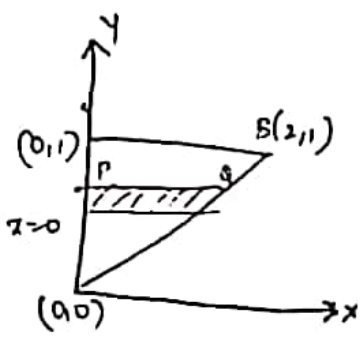
Let $f(x,y) = e^{y^2}$

The region bounded by the Δ^{le} with vertices $O(0,0)$ $A(0,1)$ $B(2,1)$

Method -②:

Eq of line OB is $y = \frac{x}{2}$

Draw a horizontal strip PQ in the region we have to fix y first.



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In the region y varies from 0 to 1.

$\therefore$ y limits are $y=0$, $y=1$

For each y x varies from a point P on y axis and $(x=0)$ to a point Q on the line $y=\frac{x}{2}$ i.e $x=2y$

$\therefore$ x limits are $x=0$, $x=2y$

$$\therefore I = \iint e^{y^2} dx\, dy = \int_{y=0}^{y=1}\left[\int_{x=0}^{x=2y} e^{y^2} dx\right] dy.$$

$$= \int_{y=0}^{y=1} e^{y^2} dy\, [x]_{x=0}^{x=2y}$$

$$= \int_{y=0}^{y=1} e^{y^2}\, 2y\, dy. = \left[e^{y^2}\right]_{y=0}^{y=1} = e^1 - e^0$$

$$= e - 1$$

Q) Evaluate $\iint (x+y)^2\, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol: Let $I = \iint_R (x+y)^2 dx\, dy$.

$f(x,y) = (x+y)^2$

The eq of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow$ which intersect x axis at the points $(-a,0)$ & $(a,0)$

It intersect y axis at point $(0,b)$ & $(0,-b)$

The ellipse is symmetric about both x & y axes.

The eq of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

we have region of integration is $-a\le x\le a$,

$-\frac{b}{a}\sqrt{a^2-x^2} \le y < \frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore \iint_R (x^2+y^2+2xy)\,dx\,dy = \int_{x=-a}^{x=a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dx\,dy.$$

$$\therefore I = \iint_R (x+y)^2\,dx\,dy = \int_{x=-a}^{x=a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy + \int_{x=-a}^{x=a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} 2xy\,dy$$

$$= 2\int_{x=-a}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx + 0.$$

[$\because$ (x^2+y^2) is even function & xy is odd function)]

$$= 2\int_{x=-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2\int_{x=-a}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$= 4\int_{x=0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

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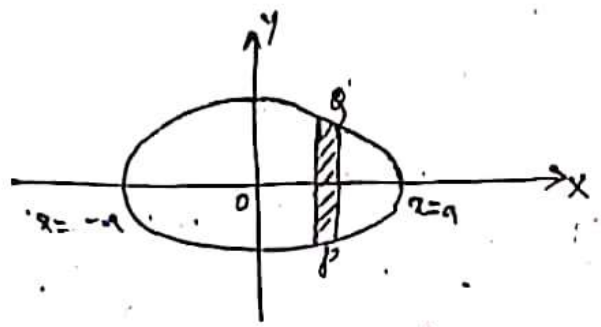
Put $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$.

Also $x=0 \Rightarrow \theta=0$ & $x=a \Rightarrow \theta = \frac{\pi}{2}$.

$$\therefore I = 4\int_0^{\pi/2}\left[\frac{b}{a}a^2\sin^2\theta\, a\cos\theta + \frac{b^3}{3a^3}a^3\cos^3\theta\right]a\cos\theta\, d\theta.$$



$$= 4\int_0^{\pi/2}\left[a^3b\sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right]d\theta$$

$$= 4\left[a^3b\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{ab^3}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}\left(a^3b + ab^3\right) = \frac{\pi}{4}ab\left(a^2+b^2\right)$$

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Area enclosed by Plane Curves.

Cartesian coordinates :

1) Consider the area enclosed by the plane curves $y = y_1(x)$, $y = y_2(x)$ and the ordinates $x = x_1$, $x = x_2$ then the area is given by, $A = \int_{x=x_1}^{x=x_2} \int_{y=y_1(x)}^{y=y_2(x)} dy\, dx.$

2) Consider the area enclosed by the curves $x = x_1(y)$, $x = x_2(y)$ and the lines $y = y_1$, $y = y_2$ then the area is given by $A = \int_{y=y_1}^{y=y_2} \int_{x=x_1(y)}^{x=x_2(y)} dx\, dy.$

Note : $\iint dx\,dy$ (or) $\iint dy\,dx$ represents area of the bounded region.

Q) Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol: W.K.T, the area of the bounded region

$$A = \iint dy\,dx.$$

G.T, the parabolas $y^2 = 4ax$ — ①. and $x^2 = 4ay$ — ②

The points of intersection of ① & ② is given by,

$$y^2 = 4ax \Rightarrow y^4 = 16a^2x^2$$

$$y^4 = 16a^2(4ay)$$

$$y(y^3 - 64a^3) = 0$$

$$y = 0,\ y = 4a$$

when $y = 0$, $x = 0$

when $y = 4a$, $x = 4a$

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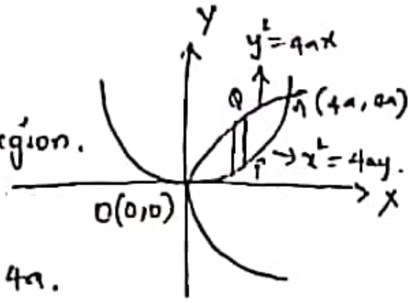
∴ The points of intersection of ① & ② is O(0,0) & A(4a, 4a).

Method -①:



Draw a vertical strip PQ in the region.
we have to fix x first in the region x varies from 0 to 4a.

∴ x limits are $x=0$, $x=4a$.

For each x, y varies from a point P on parabola $y=\frac{x^2}{4a}$ to a point Q on the parabola $y^2=4ax$.

i.e $y=2\sqrt{ax}$

∴ Area $A=\iint dy\,dx = \int_{x=0}^{x=4a}\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dy\,dx.$

$$= \int_{x=0}^{x=4a} dx\,\Big[y\Big]_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}}$$

$$= \int_{x=0}^{x=4a}\left[2\sqrt{a}\sqrt{x}-\frac{x^2}{4a}\right]dx$$

$$= \left[2\sqrt{a}\,\frac{2}{3}\,x^{3/2}-\frac{x^3}{12a}\right]_{x=0}^{x=4a}$$

$$= \left[\frac{4}{3}\sqrt{a}\,(4a)^{3/2}-\frac{(4a)^3}{12a}\right]-0$$

$$= \frac{4}{3}\sqrt{a}\cdot 4\sqrt{2}\cdot a\sqrt{2}-\frac{64a^3}{12a}$$

$$= \frac{16a^2(2)}{3}-\frac{16a^2}{3} = \frac{32a^2-16a^2}{3} = \frac{16a^2}{3}$$

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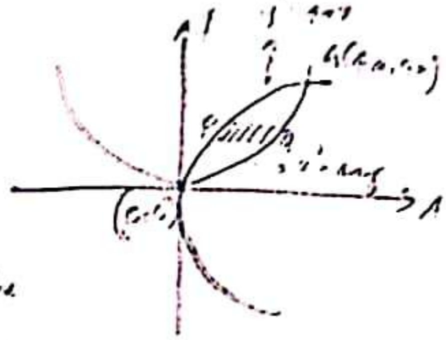
Method - ②



Draw a horizontal strip PQ, in the region.

we have to fix y first in the region y varies from 0 to 4a

$\therefore$ y limits are $y=0$, $y=4a$.

For each y, x varies from a point P on the parabola $x=\frac{y^2}{4a}$ to a point Q on the parabola $x^2=4ay$

i.e. $x = 2\sqrt{ay}$.

$\therefore$ x limits are $x=\frac{y^2}{4a}$, $x=2\sqrt{ay}$.

$$\text{Area } A = \int_{y=0}^{y=4a}\int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} dx\,dy.$$

$$= \int_{y=0}^{y=4a} dy\,[x]_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}}$$

$$= \int_{y=0}^{y=4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{\frac{3}{2}} - \frac{1}{4a}\frac{y^3}{3}\right]_{y=0}^{y=4a}$$

$$= \left[\frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{1}{4a}\cdot\frac{\cancel{64}^{16}a^3}{3}\right]$$

$$= \frac{4\sqrt{a}}{3}(4a)(2\sqrt{a}) - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

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6) Find the area of the region bounded by the curves $y = x^2$ and $y = x$.

Sol: W.K.T, the area of the bounded region

$$A = \iint dy\,dx.$$

G.T, the curves $y = x^2$ — ① and $y = x$ — ②.

The points of intersection of ① & ② is given by,

$$y = x^2 \Rightarrow \cancel{y^2 = x^2} \quad y = y^2$$

$$y - y^2 = 0$$

$$y(y-1) = 0$$

$$y = 0,\ y = 1$$

when $y = 0$, $x = 0$

when $y = 1$, $x = 1$

∴ The points of intersection of ①&② is $O(0,0)$ & $A(1,1)$

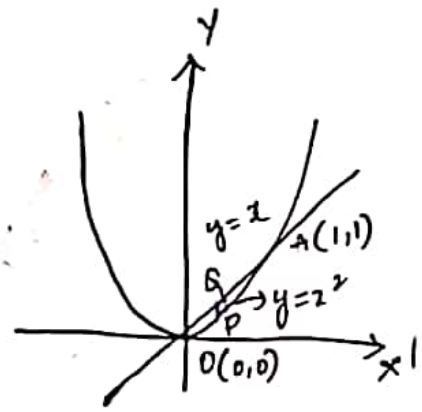
Method -①:

Draw a Vertical Strip PQ in the region.

we have to fix x first in the region : x varies from 0 to 1.

∴ x limits are $x = 0$, $x = 1$.

For each x, y varies from a point P on the curve $y = x^2$ to a point Q on the curve $y = x$.



∴ Area: $A = \iint dy\,dx$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} dy\,dx.$$

$$= \int_{x=0}^{x=1} dx\,[y]_{x^2}^{x}$$

$$= \int_{x=0}^{x=1} (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{x=1}$$

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$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}.$$

Method - 2 :

Draw a horizontal strip PQ in the region.

We have to fix y first in the region y varies from 0 to 1.

$\therefore$ y limits are $y=0$, $y=1$.

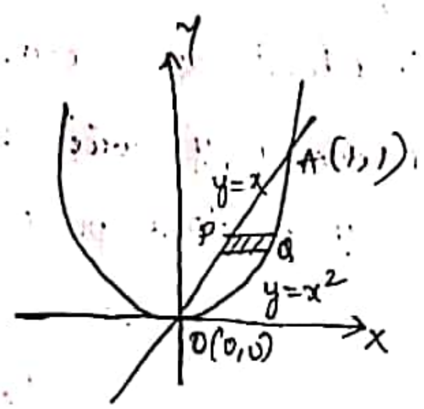
For each y, x varies from point P on the curve $y=x$ to a point Q on the curve $x=\sqrt{y}$.

$\therefore$ x limits are $x=y$, $x=\sqrt{y}$.

$$\text{Area } A = \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} dx\, dy$$

$$= \int_{y=0}^{y=1} dy \, [x]_{y}^{\sqrt{y}}$$

$$= \int_{y=0}^{y=1} [\sqrt{y} - y]\, dy$$



$$= \left[\frac{y^{3/2}}{3/2} - \frac{y^2}{2} \right]_0^1$$

$$= \left[\frac{2}{3} y^{3/2} - \frac{y^2}{2} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{2\times 2 - 1\times 3}{6}$$

$$= \frac{4-3}{6} = \frac{1}{6}$$

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Q) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using double integral.

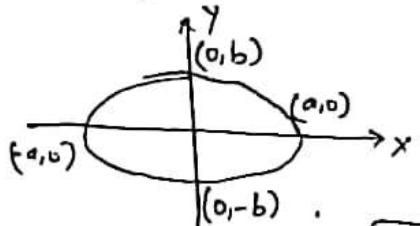
Sol: G.T, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which intersect x axis at the points $(-a, 0)$ and $(a, 0)$.

It intersect y axis at the points $(0, b)$ & $(0, -b)$.

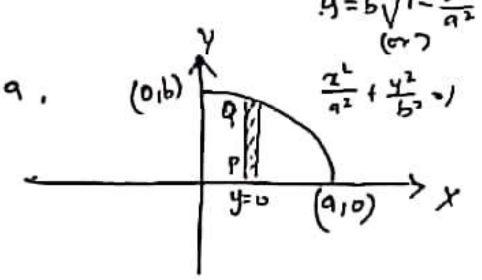
An ellipse is symmetric about both x & y axes.

Method - ① :

Draw a vertical strip PQ in the region.



we have to fix x first.



In the region x varies from 0 to a.

$\therefore$ x limits are $x = 0$, $x = a$

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For each x, y varies from a point P on x axis $(y=0)$ to a point Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

i.e. $y = b\sqrt{1-\frac{x^2}{a^2}}$.

$\therefore$ y limits are $y=0$, $y=b\sqrt{1-\frac{x^2}{a^2}}$.

Area $A = \iint dy\,dx$

Area of ellipse $A = 4\iint dy\,dx$

$$A = 4\int_{x=0}^{x=a}\int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$

$$A = 4\int_{x=0}^{x=a} [y]_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 4\int_{x=0}^{x=a} b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$= \frac{4b}{a}\int_{x=0}^{x=a}\sqrt{a^2-x^2}\,dx$$

put $x = a\cos\theta$

$dx = -a\sin\theta\,d\theta$

when $x=0$, $\theta=\frac{\pi}{2}$

when $x=a$, $\theta=0$

$$= \frac{4b}{a}\int_{\theta=\frac{\pi}{2}}^{\theta=0}\sqrt{a^2-a^2\cos^2\theta}\,(-a\sin\theta)\,d\theta$$

$$= \frac{-4ba^2}{a}\int_{\pi/2}^{0}\sin^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = 2ab\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2ab\left(\frac{\pi}{2}-0\right) = \pi ab.$$

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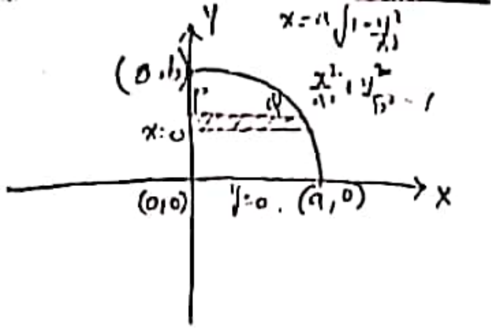
Method - ②



Draw a horizontal strip PQ in the region.

we have to fix y first.

In the region y varies from '0' to 'b'.

$\therefore$ y limits are $y=0$, $y=b$

For each 'y', x varies from a point 'P' on y-axis to point Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$x = a\sqrt{1-\frac{y^2}{b^2}}$$

$\therefore$ x limits are $x=0$, $x = a\sqrt{1-\frac{y^2}{b^2}}$

Area $A = \iint dy\,dx$.

Area of ellipse $A = 4\iint dx\,dy$

$$A = 4\int_{y=0}^{y=b}\int_{x=0}^{x=a\sqrt{1-\frac{y^2}{b^2}}} dx\,dy.$$

$$= 4\int_{y=0}^{y=b} dy\,[x]_0^{a\sqrt{1-\frac{y^2}{b^2}}}$$

$$= 4\int_{y=0}^{y=b} a\sqrt{1-\frac{y^2}{b^2}}\,dy.$$

put $y = b\cos\theta$.
$dy = -b\sin\theta\,d\theta$
when $y=0$, $\theta=0$
$y=b$, $\theta=\pi/2$

$$= \frac{4a}{b}\int_0^b \sqrt{b^2-y^2}\,dy$$

$$= -\frac{4a}{b}\int_0^b \sqrt{b^2-b^2\cos^2\theta}\; b\sin\theta\,d\theta$$

$$= \frac{-4ab^2}{b}\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 4ab\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = 2ab\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta = 2ab\left(\frac{\pi}{2}-0\right) = \pi ab$$

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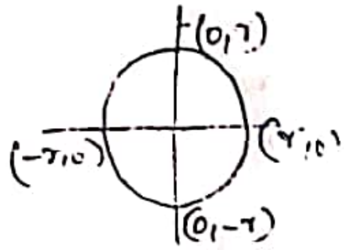
Q) Using double integral find the area of the circle $x^2+y^2=r^2$

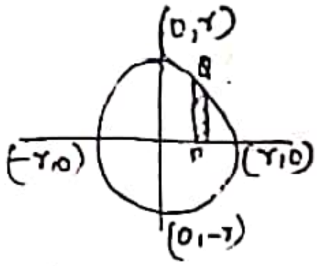
Sol: G.T, $x^2+y^2=r^2$ which intersect x-axis at $(-r,0)$ & $(r,0)$

y-axis at $(0,r)$ & $(0,-r)$



Circle is symmetrical about x and y axis.

Method - ①:



Draw a vertical strip in region.

Fix x first, x varies from 0 to r. limits of x are $x=0$, $x=r$.

For each 'x', y varies from a point P on x axis & to the point Q on the circle. $x^2+y^2=r^2$.

i.e $y=\sqrt{r^2-x^2}$

$\therefore$ y limits are $y=0$, $y=\sqrt{r^2-x^2}$

Area $A=\iint dy\,dx \Rightarrow 4\iint dy\,dx$

$$=4\int_{x=0}^{x=r}\int_{y=0}^{y=\sqrt{r^2-x^2}} dy\,dx.$$

$$=4\int_{x=0}^{x=r}(y)_0^{\sqrt{r^2-x^2}}\,dx.$$

$$=4\int_{x=0}^{x=r}\sqrt{r^2-x^2}\,dx$$

put $x=r\cos\theta$

$dx=-r\sin\theta\,d\theta$

when $x=0$

$\theta=\pi/2$

when $x=r$

$\theta=0$

$$=4\int_{\theta=\pi/2}^{\theta=0}\sqrt{r^2-r^2\cos^2\theta}\,(-r\sin\theta)\,d\theta$$

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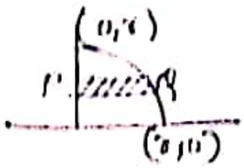
$$= 4\int_{\theta=\pi/2}^{\theta=0} r\sin\theta\,(-r\sin\theta)\,d\theta = -4r^2\int_{\theta=\pi/2}^{\theta=0}\sin^2\theta\,d\theta$$

$$= -4r^2\int_{\theta=\pi/2}^{\theta=0}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -\frac{4r^2}{2}\left[(\theta)_{\pi/2}^{0} - \frac{1}{2}(\sin 2\theta)_{\pi/2}^{0}\right] = -2r^2\left[-\frac{\pi}{2} - \frac{1}{2}(0-0)\right] = \pi r^2$$

<u>Method - ②</u>

Draw a horizontal strip PQ.



we have to fix 'y' first, y varies from 0 to r. y limits are $y=0$, $y=r$.

For each 'y', x varies from a point P on y axis to point Q on the circle $x^2+y^2=r^2$.

x limits are $x=0$, $x=\sqrt{r^2-y^2}$

$$A = 4\int_{y=0}^{y=r}\int_{x=0}^{x=\sqrt{r^2-y^2}} dx\,dy$$

put $y = r\sin\theta$

$dy = r\cos\theta\,d\theta$

when $y=0,\ \theta=0$

$y=r,\ \theta=\pi/2$

$$= 4\int_{y=0}^{y=r}\left(\sqrt{r^2-y^2}\right)dy$$

$$= 4\int_{\theta=0}^{\theta=\pi/2}\left(\sqrt{r^2-r^2\sin^2\theta}\right)r\cos\theta\,d\theta$$

$$= 4\int_{\theta=0}^{\theta=\pi/2} r^2\cos^2\theta\,d\theta$$

$$= 4r^2\int_{\theta=0}^{\theta=\pi/2}\left(\frac{1-\sin 2\theta}{2}\right)$$

$$= \pi r^2 \quad //$$

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Change of Order of Integration :-

(i) Working Rule for change of order of integration for $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$

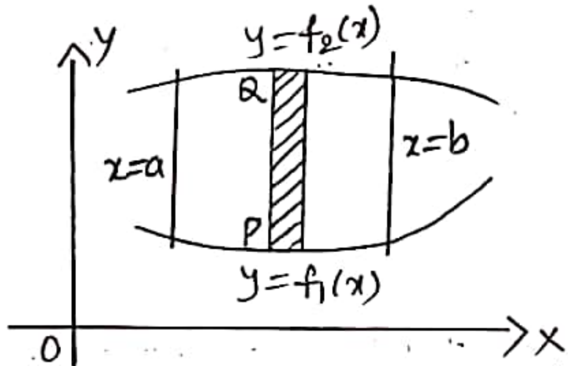
Step 1 :- In the given integral, the inner integral limits are in terms of x, so these are limits of y.

$\therefore$ y limits are $y=f_1(x)$, $y=f_2(x)$.

The outer integral limits are constants so these are limits of x.

$\therefore$ x limits are $x=a$, $x=b$.

Step 2 :- Draw the region of integration by drawing the curves $y=f_1(x)$, $y=f_2(x)$ and the lines $x=a$, $x=b$.



Step 3 :- To change the order of integration we draw a horizontal strip PQ in the region.

In this case we have to fix y first (i.e y limits are constants)

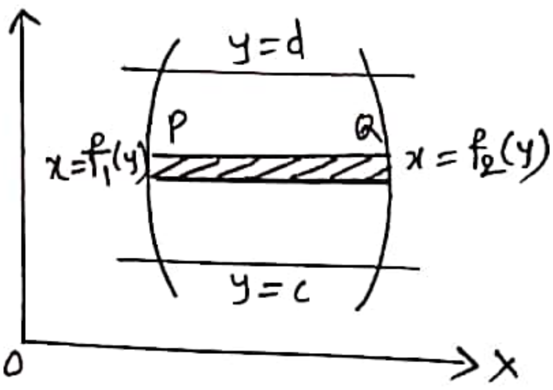
Suppose in the region y varies from c to d.

$\therefore$ y limits are $y=c$, $y=d$.

For each y, x varies from a point P on the curve $x=f_1(y)$ to a point Q on the curve $x=f_2(y)$

$\therefore$ x limits are $x=f_1(y)$, $x=f_2(y)$.

$$\therefore I = \int_{x=a}^{x=b} \int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\,dx = \int_{y=c}^{y=d} \int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\,dy$$



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(ii) Working Rule for change of order of integration for $\int_{a}^{b}\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$:-

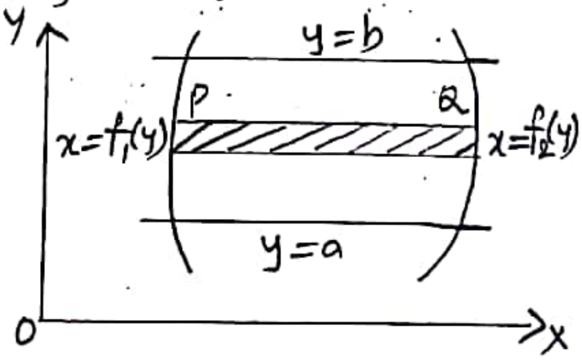
Step 1 :- In the given Integral, the inner integral limits are in terms of y, so these are limits of x.

$\therefore$ x limits are $x = f_1(y)$, $x = f_2(y)$.

The outer integral limits are constants so these are limits of y.

$\therefore$ y limits are $y = a$, $y = b$.

Step 2 :- Draw the region of integration by drawing the curves $x = f_1(y)$, $x = f_2(y)$ and the lines $y = a$, $y = b$.



Step 3 :- To change the order of integration. we draw a vertical strip PQ in the region.

In this case we have to fix x first (i.e x limits are constants)

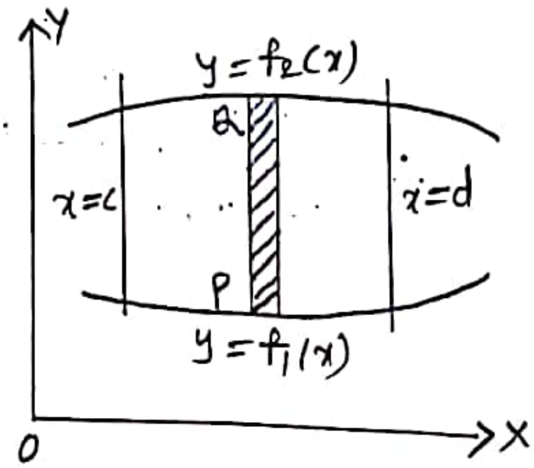
Suppose in the region x varies from c to d.

$\therefore$ x limits are $x = c$, $x = d$.

For each x, y varies from a point P on the curve $y = f_1(x)$ to a point Q on the curve $y = f_2(x)$.

$\therefore$ y limits are $y = f_1(x)$, $y = f_2(x)$.

$$\therefore I = \int_{y=a}^{y=b}\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx\,dy = \int_{x=c}^{x=d}\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy\,dx.$$



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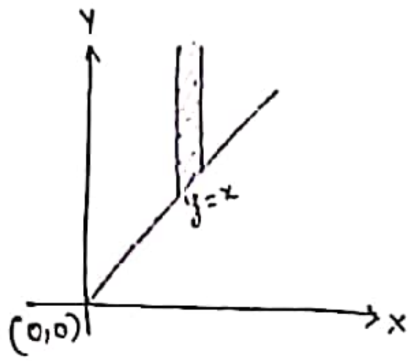
Q) Evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$. by changing the order of integration.

Sol: Let $I = \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$.

Let $f(x,y) = \frac{e^{-y}}{y}$



In the given integral the inner integral limits are in terms of x.

$\therefore$ these are limits of y.

$\therefore$ y limits are $y=x$, $y\to\infty$.

The outer integral limits are constants these are limits of x.

$\therefore$ x limits are $x=0$, $x\to\infty$

$\therefore$ The region of integration is above the line $y=x$.

To change the order of integration draw a horizontal strip PQ in the region.

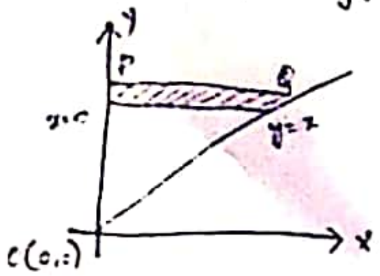
We have to fix y first.

In the region y varies from 0 to ∞.

$\therefore$ y limits are $y=0$, $y\to\infty$.

For each y, x varies from a point P on y axis $(x=0)$ to a point Q on the line $x=y$.

$\therefore$ x limits are $x=0$, $x=y$.



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$$\therefore\ I = \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx = \int_{y=0}^{y\to\infty}\int_{x=0}^{x=y} \frac{e^{-y}}{y}\,dx\,dy.$$

$$= \int_{y=0}^{y\to\infty} \frac{e^{-y}}{y}\,dy\ [x]_{x=0}^{x=y}$$

$$= \int_{y=0}^{y\to\infty} \frac{e^{-y}}{y}\,dy\ [y-0]$$

$$= \int_{y=0}^{y\to\infty} e^{-y}\,dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= \left[\frac{e^{-\infty}}{-1} - \left(\frac{e^{-0}}{-1}\right)\right] = 1$$

Q) Evaluate $\int_0^{\infty}\int_0^{x} x e^{-x^2/y}\,dy\,dx$ by changing the order of integration.

Sol: Let $I = \int_0^{\infty}\int_0^{x} x e^{-x^2/y}\,dy\,dx$

Let $f(x,y) = x e^{-x^2/y}$

In the given integral the inner integral limits in terms of 'x'.

So, These are limits of 'y'.

y limits are $y=0,\ y=x$.

The outer integral limits are constants,

These are limits are constants.

$\therefore$ x limits are $x=0,\ x\to\infty$.

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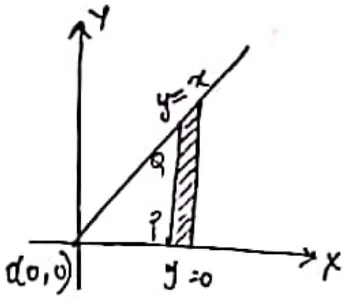
The region of integration is in the first quadrant that lies to the right side of the line $y = x$.



To change the order of integration draw a horizontal strip in the region.

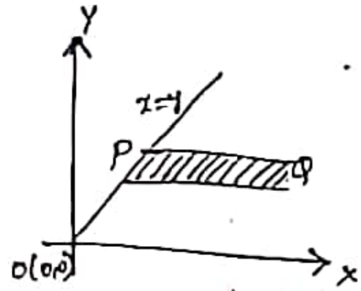
In the region y varies from 0 to ∞.

∴ y limits are $y = 0$, $y \to \infty$

For each y, x varies from a point P on the line $x = y$ to ∞.

∴ x limits are $x = y$, $x \to \infty$.

$$I = \int_0^\infty \int_0^x x e^{-x^2/y} \, dy \, dx$$



$$= \int_{y=0}^{y\to\infty} \int_{x=y}^{x\to\infty} x e^{-x^2/y} \, dx \, dy$$

put $\frac{x^2}{y} = t$

$x\,dx = \frac{y}{2}\,dt$

when $x = y, t = y$

when $x \to \infty, t \to \infty$

$$\int_{y=0}^{y\to\infty} \int_{t=y}^{t\to\infty} \frac{y}{2} e^{-t} \, dt \, dy$$

$$= \frac{1}{2} \int_{y=0}^{y\to\infty} y \, dy \left(\frac{e^{-t}}{-1}\right)_{t=y}^{t\to\infty}$$

$$= \frac{-1}{2} \int_{y=0}^{y\to\infty} y \, dy \left[e^{-\infty} - e^{-y}\right]$$

$$= \frac{1}{2} \int_{y=0}^{y\to\infty} y e^{-y} \, dy$$

$$= \frac{1}{2} \left[y \left(\frac{e^{-y}}{-1}\right) - 1 \left(\frac{e^{-y}}{(-1)^2}\right) \right]_0^\infty$$

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$$= \frac{1}{2}\left[\frac{-y}{e^y} - \frac{1}{e^y}\right]_0^\infty$$

$$= \frac{1}{2}\left[(0) - (-1)\right] = \frac{1}{2}$$

Q) By changing the order of integration, Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$.

Sol: Let $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

Let $f(x,y) = xy$

In the given integral the inner integral limits are in terms of 'x'. So these are limits of 'y'.

$\therefore$ y limits are $y = x^2$, $y = 2-x$.

The outer integral limits are constants. These are limits of 'x'.

$\therefore$ 'x' limits are $x=0, x=1$.

we have $y = x^2$ — ① is a parabola.

$y = 2-x$ — ② is a straight line.

The point of intersection of ① & ② is given by,

$\cancel{x^2 = 2x}$ $x^2 = 2-x$

$x^2 + x - 2 = 0$

$x = -2, x = 1$

when $x = -2$ $x = 1$

$y = 4$ $y = 1$

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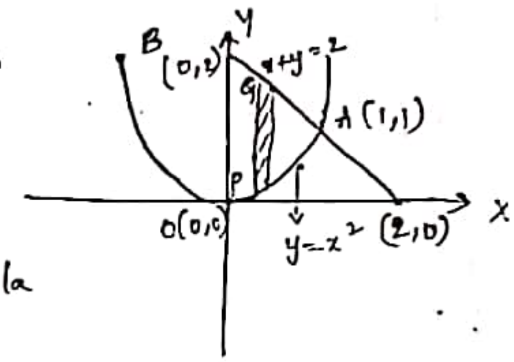
∴ The points of intersection of ① & ② is $(-2,4)$ & $(1,1)$.

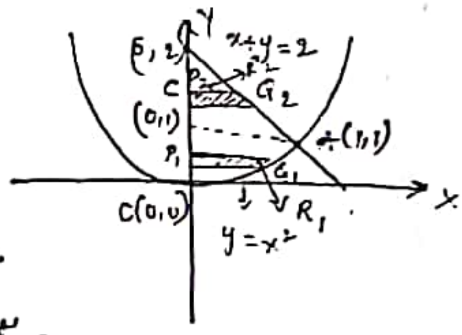
∴ The region of integration is OABO



The region of integration is bounded by y-axis, the parabola $y=x^2$ & the line $x+y=2$.

Divide the region R into 2 parts R_1 (Area OAC) and R_2 (Area CAB)

Area OAB = Area OAC + Area CAB.



To change the order of integration.

Region R_1 :

Draw a horizontal strip P_1, Q_1.

We have to fix y first.

The region y varies from 0 to 1.

∴ y limits are $y=0$, $y=1$

For each y, x varies from a point P_1 on y axis $(x=0)$ to a point Q_1 on the parabola $y=x^2$.

∴ x limits are $x=0$, $x=\sqrt{y}$.

Region R_2 :

Draw a horizontal strip P_2, Q_2 in the region R_2.

We have to fix y first.

In the region y varies from 1 to 2.

∴ y limits are $y=1$, $y=2$.

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For each y, x varies from a point P_2 on y-axis $(x=0)$ to a point Q_2 on the line $x+y=2$.

i.e $x = 2-y$.

$\therefore$ x limits are $x=0$, $x=2-y$.

$I = I_1 + I_2$

$$\int_0^1 \int_{x^2}^{2-y} xy\,dx\,dy = \iint_{OAC} xy\,dx\,dy + \iint_{CAB} xy\,dx\,dy.$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{y=2} \int_{x=0}^{x=2-y} xy\,dx\,dy.$$

$$= \int_{y=0}^{y=1} y\,dy \left[\frac{x^2}{2}\right]_{x=0}^{x=\sqrt{y}} + \int_{y=1}^{y=2} y\,dy \left[\frac{x^2}{2}\right]_{x=0}^{x=2-y}.$$

$$= \frac{1}{2}\int_{y=0}^{y=1} y\,dy\left[(\sqrt{y})^2 - 0\right] + \frac{1}{2}\int_{y=1}^{y=2} y\,dy\left[(2-y)^2 - 0\right]$$

$$= \frac{1}{2}\int_{y=0}^{y=1} y^2\,dy + \frac{1}{2}\int_{y=1}^{y=2}\left[y^3 - 4y^2 + 4y\right]dy.$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_{y=0}^{y=1} + \frac{1}{2}\left[\frac{y^4}{4} - \frac{4y^3}{3} + 2y^2\right]_{y=1}^{y=2}$$

$$= \frac{1}{2}\left[\frac{1}{3} - 0\right] + \frac{1}{2}\left[\left\{4 - \frac{32}{3} + 8\right\} - \left\{\frac{1}{4} - \frac{4}{3} + 2\right\}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(12 - \frac{32}{3}\right) - \left(\frac{1}{4} - \frac{10}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{36-32}{3} - \frac{3-40}{12}\right] = \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3} + \frac{37}{12}\right] \qquad = \frac{3}{8}$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{4\times 4 + 37}{12}\right] = \frac{1}{6} + \frac{1}{2}\left(\frac{53}{12}\right) = \frac{1}{6} + \frac{53}{24} = \frac{4+53}{24}$$

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8) Evaluate $\int_3^5 \int_0^{4/x} xy\,dy\,dx$ by changing the order of integration.

Sol: Let $I = \int_3^5 \int_0^{4/x} xy\,dy\,dx$.

$f(x,y) = xy$.

In the given integral the inner integral limits are in terms of x. So they are limits of y.

$\therefore$ y limits are $y=0$ & $y=\frac{4}{x}$.

The outer integral limits are constants, these are limits of x.

$\therefore$ x limits are $x=3$, $x=5$.

Here x varies from 3 to 5, for each x, y varies from $y=0$ to $y=\frac{4}{x}$.

The region of integration is bounded b/w the lines $x=3$, $x=5$, x-axis and the curve $y=\frac{4}{x}$.

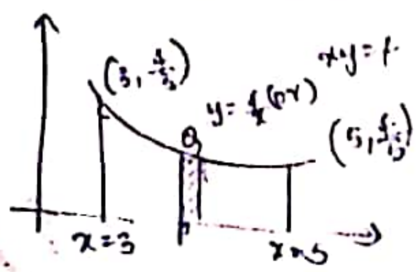
The point of intersection of the line $x=3$ & the curve $y=\frac{4}{x}$ is $(3, \frac{4}{3})$

The point of intersection of the line $x=5$ & the curve $y=\frac{4}{x}$ is $(5, \frac{4}{5})$.

To change the order of integration.

Divide the region into two parts R_1 (Area ABCD) and R_2 (Area DCE)

Area ABCE = Area ABCD + Area DCE.



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i) Draw a horizontal strip P_1Q_1 in the region R_1.

we fix y first

In the region R_1, y varies from 0 to $\frac{4}{5}$.

$\therefore$ y limits are $y=0$, $y=\frac{4}{5}$.

For each y, x varies from a point P_1 on the line $x=3$ to the point Q_1 on the line $x=5$.

$\therefore$ x limits are $x=3$, $x=5$.

ii) Draw a horizontal P_2Q_2 in the region R_2, we fix y first.

In the region R_2, y varies from $y=\frac{4}{5}$ to $y=\frac{4}{3}$.

$\therefore$ y limits are $y=\frac{4}{5}$, $y=\frac{4}{3}$

For each y, x varies from a point P_2 on the line $x=3$ to the point Q_2 on the curve $xy=4$ i.e $x=\frac{4}{y}$

$\therefore$ x limits are $x=3$, $x=\frac{4}{y}$.

$$\int_3^5\int_0^{4/x} xy\,dy\,dx = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy$$

$$= \int_{y=0}^{y=\frac{4}{5}x}\int_{x=3}^{x=5} xy\,dx\,dy + \int_{y=\frac{4}{5}}^{y=4/3}\int_{x=3}^{x=\frac{4}{y}} xy\,dx\,dy.$$

$$= \int_{y=0}^{y=\frac{4}{5}}\int_{x=3}^{x=5} xy\,dx\,dy + \int_{y=\frac{4}{5}}^{y=\frac{4}{3}}\int_{x=3}^{x=\frac{4}{y}} xy\,dx\,dy.$$

$$= \left[\int_{y=0}^{y=\frac{4}{5}} y\,dy \int_{x=3}^{x=5} x\,dx\right] + \int_{y=\frac{4}{5}}^{y=\frac{4}{3}} y\,dy\left(\frac{x^2}{2}\right)_{x=3}^{x=\frac{4}{y}}$$

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$$= \left(\frac{y^2}{2}\right)_0^{4/5} \left(\frac{x^2}{2}\right)_3^5 + \int_{y=4/5}^{y=4/3} y\,dy \left(\frac{16}{2y^2} - \frac{9}{2}\right)$$

$$= \left(\frac{16}{50}\right)(8) + \left[8\log|y| - \frac{9}{4}y^2\right]_{4/5}^{4/3}$$

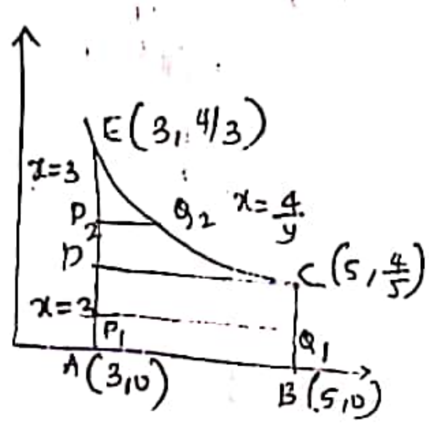
$$= \frac{64}{25} + \left[\left\{8\log\left|\frac{4}{3}\right| - \frac{9}{4}\cdot\frac{16}{9}\right\} - \left\{8\log\left|\frac{4}{5}\right| - \frac{9}{4}\,\frac{16}{25}\right\}\right]$$

$$= \frac{64}{25} + \left[8\log 4 - 8\log 3 - 4 - 8\log 4 + 8\log 5 + \frac{36}{25}\right]$$

$$= \frac{64}{25} + \left[8\log 5 - 8\log 3 - \frac{64}{25}\right]$$

$$= 8\log_e 5 - 8\log_e 3$$

$$= 8\log_e\left(\frac{5}{3}\right)$$



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Change of Variables in double integral.

Transformation of Coordinates :-

If $x = f(u,v), y = g(u,v)$ are the relation b/w old variables (x,y) & new variables (u,v) of the new coordinate system.

Then $\iint_R f(x,y)\,dx\,dy = \iint_R F(u,v)\,|J|\,du\,dv.$

where $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

which is called Jacobian of the coordinate transformation.

Q) Evaluate $\iint_R (x+y)^2\,dx\,dy$ where R is the parallelogram in xy plane with vertices $(1,0)(3,1)$ $(2,2)$ and $(0,1)$ by using transformations $u = x+y$ and $v = x-2y$.

Sol: Let $I = \iint_R (x+y)^2\,dx\,dy$.

Let $f(x,y) = (x+y)^2$

G.T, R is the parallelogram with vertices $A(1,0)$, $B(0,1)$ $C(2,2)$ $D(3,1)$

G.T, the transformations $u = x+y$ —① & $v = x-2y$ —②

we have

$A(1,0)$	$u=1$	$v=1$	— $P(1,1)$
$B(0,1)$	$u=1$	$v=-2$	— $Q(1,-2)$
$C(2,2)$	$u=4$	$v=-2$	— $R(4,-2)$
$D(3,1)$	$u=4$	$v=1$	— $S(4,1)$.

Solving ① & ②, we get

$x = \frac{1}{3}(2u+v)$, $y = \frac{1}{3}(u-v)$.

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We have $\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv$ —— ③.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

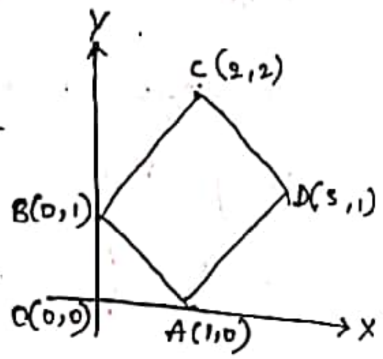
$$\frac{\partial x}{\partial u} = \frac{2}{3} \qquad \frac{\partial x}{\partial v} = \frac{1}{3}$$

$$\frac{\partial y}{\partial u} = \frac{1}{3} \qquad \frac{\partial y}{\partial v} = \frac{-1}{3}$$

$$J = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = \frac{-2}{9} - \frac{1}{9} = \frac{-1}{3}$$

$$|J| = \frac{1}{3}$$

The parallelogram ABCD in XY-plane is transformed to square PQRS in UV plane.

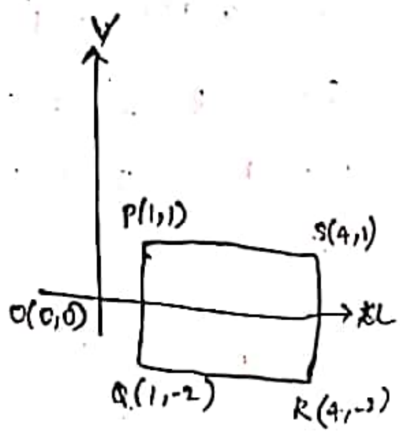


In the region u varies from 1 to 4.

∴ u limits are $u=1, u=4$

V varies from -2 to 1.

∴ V limits are $V=-2, V=1$



From ③,

$$\iint_R (x+y)^2\,dx\,dy = \iint_R u^2 \frac{1}{3}\,du\,dv.$$

$$= \frac{1}{3}\int_{u=1}^{u=4}\int_{v=-2}^{v=1} u^2\,du\,dv.$$

$$= \frac{1}{3}\int_{u=1}^{u=4} u^2\,du \int_{v=-2}^{v=1} dv$$

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$$= \frac{1}{3}\left[\frac{u^3}{3}\right]_1^4 \left[v\right]_{-2}^{1}$$

$$= \frac{1}{9}\left[4^3-1\right]\left[1-(-2)\right] = \frac{\cancel{63}^{\,\cancel{21}\,7}}{\cancel{9}_{\cancel{3}}}(3) = 21$$

Q) By using the transformations $x+y=u$, $y=uv$.

S.T $\displaystyle\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dx\,dy = \frac{e-1}{2}$.

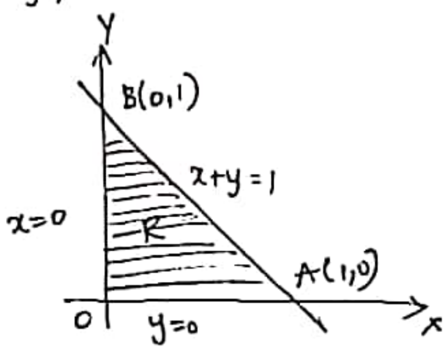
Sol: Let $I = \displaystyle\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dx\,dy$.

Let $f(x,y) = e^{\frac{y}{x+y}}$

The region of integration is given by,

$y=0$, $y=1-x$, $x=0$ and $x=1$.

i.e the Δ^{le} OAB.



Given transformation is

$x+y=u$ —①

$y=uv$ —②

we have $A(1,0)$ $u=1$ $v=1$ — $P(1,1)$

$B(0,1)$ $u=1$ $v=1$ — $Q(1,1)$

$C(0,0)$ $u=0$ $v=0$ — $R(0,0)$

∴ The solving ① & ②

$x = u(1-v)$, $y = uv$.

∴ The region R is transformed to R' where R' is the Square OPQR in the uv plane. The Jacobian of transformation is given by.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

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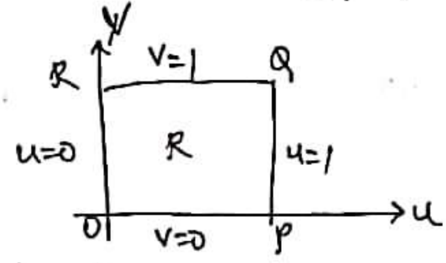
$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u - uv + uv = u \quad \Rightarrow |J| = u$$



So the region u varies from 0 to 1,

$\therefore$ u limits are $u=0, u=1$

V varies from 0 to 1

$\therefore$ V limits $v=0, v=1$

Hence $\displaystyle\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy\, dx$

$$= \iint_{R'} e^{uv/u} |J|\, du\, dv$$

$$= \int_{v=0}^{1} \int_{u=0}^{1} e^{v} u \, du\, dv = \int_{v=0}^{1} e^{v} \left(\frac{u^2}{2}\right)_0^1 dv.$$

$$= \frac{1}{2}\int_0^1 e^{v} dv = \frac{1}{2}\left(e^{v}\right)_0^1$$

$$= \frac{1}{2}(e-1).$$

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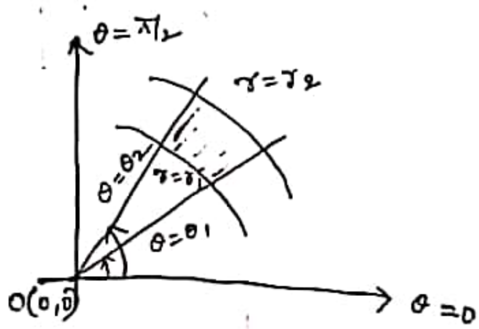
Polar Coordinates in double integrals :-

Q) Evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$. Over the region bounded by the lines $\theta=\theta_1$, $\theta=\theta_2$ and the curves $r=r_1$, $r=r_2$.

First we integrate w.r.t r b/w the limits $r=r_1$ & $r=r_2$ keeping θ fixed.

The resulting expression is integrated w.r.t 'θ from θ_1 to θ_2.

In this integral r_1, r_2 are functions of θ and θ_1, θ_2 are constants.



Q) Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

Sol: $I=\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

In the given integral the inner 'integral limits are in terms of θ.

So these are limits of r.

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$\therefore$ r limits $r=0$, $r=a\sin\theta$

The outer integral limits are constants these are limits of θ.

$\therefore$ θ limits $\theta=0$, $\theta=\pi$.

First integrate w.r.t r keeping θ fixed and then integrate w.r.t θ.

$$I = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a\sin\theta} r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\theta=\pi} d\theta \left[\frac{r^2}{2}\right]_{r=0}^{r=a\sin\theta}.$$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=\pi} a^2\sin^2\theta\, d\theta.$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\theta=\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta.$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[\left\{\pi - \frac{\sin 2\pi}{2}\right\} - 0\right]$$

$$= \frac{a^2\pi}{4}$$

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Q) Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,dr\,d\theta$.

Sol: Let $I = \int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,dr\,d\theta$.

Let $f(r,\theta) = e^{-r^2} r$

Here all the four limits are constants.

θ limits are 0 to $\pi/2$.

r limits are 0 to ∞.

$$\therefore \quad I = \int_{r=0}^{r\to\infty}\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-r^2} r\,dr\,d\theta.$$

$$= \int_{r=0}^{r\to\infty} \{ e^{-r^2} r\,dr \int_{\theta=0}^{\theta=\pi/2} d\theta.$$

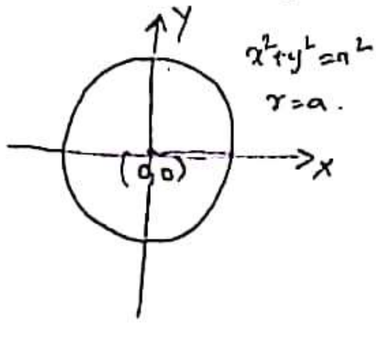
$$= \frac{-1}{2}\int_{r=0}^{r\to\infty} e^{-r^2}(-2r)\,dr\,[\theta]_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{-1}{2}\left[e^{-r^2}\right]_{r=0}^{r\to\infty}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{-1}{2}\left[e^{-\infty}-e^{0}\right]\frac{\pi}{2} = \frac{\pi}{4}$$

Note:

i) Eq of the circle $C(0,0)$ & radius 'a' units is $x^2+y^2=a^2$ and the same eq in polar coordinates is $r=a$.



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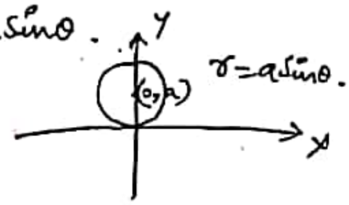
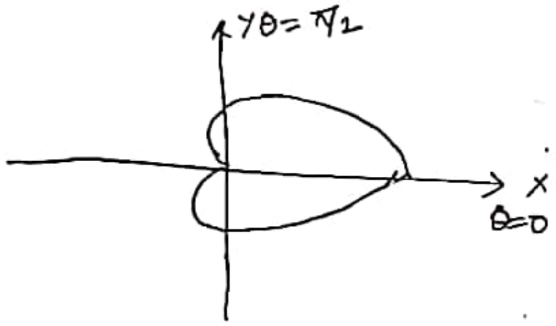
ii) Eq of the circle centre on x-axis passes through the origin is $x^2+y^2-2ax=0$ and the same in polar coordinates $r=a\cos\theta$.

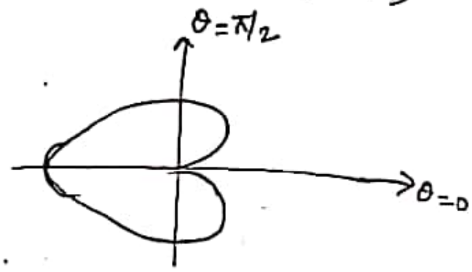
iii) Eq of the circle centre on y-axis and passes through the origin is $x^2+y^2-2ay=0$ and the same in polar coordinates $r=a\sin\theta$.



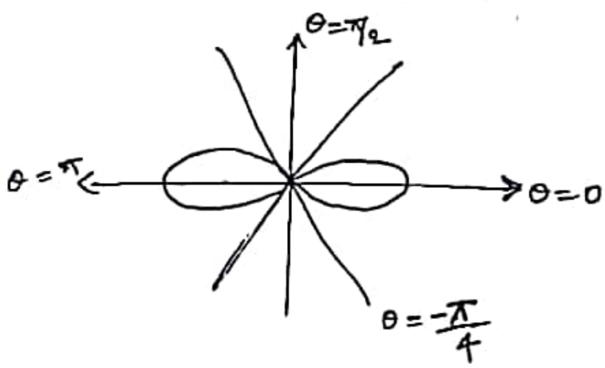
iv) Cardioid $r=a(1+\cos\theta)$



Cardioid $r=a(1-\cos\theta)$



Lemniscate $r^2=a^2\cos 2\theta$.



Q) Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardioid $r=a(1-\cos\theta)$ above the initial line.

Sol: Let $I=\iint r\sin\theta\, dr\, d\theta$.

Let $f(r,\theta)=r\sin\theta$.

G.T, the cardioid $r=a(1-\cos\theta)$

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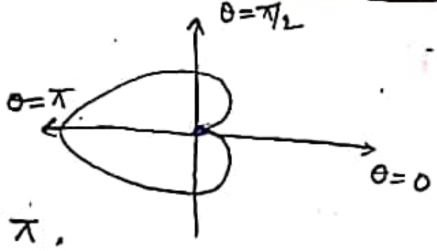
The region of integration is above the initial line $\theta = 0$ (x-axis)



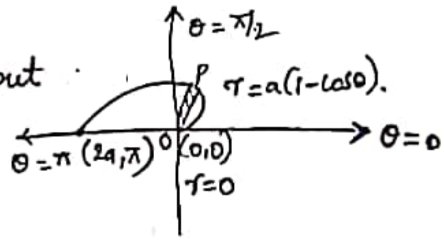
In the region θ varies from 0 to π.

$\therefore$ θ limits $\theta = 0$, $\theta = \pi$

Draw a radius vector OP in the region which start at O ($r = 0$) and terminates at P ($r = a(1-\cos\theta)$)

$\therefore$ limits of r is $r = 0$, $r = a(1-\cos\theta)$.

[The cardioid is symmetry about initial line. It passes through the Origin].



$$I = \iint r\sin\theta \, dr \, d\theta.$$

$$I = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1-\cos\theta)} r\sin\theta \, dr \, d\theta.$$

$$= \int_{\theta=0}^{\theta=\pi} \sin\theta \, d\theta \left[\frac{r^2}{2}\right]_{r=0}^{r=a(1-\cos\theta)}$$

$$= \frac{1}{2} \int_{\theta=0}^{\theta=\pi} a^2(1-\cos\theta)^2 \sin\theta \, d\theta.$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\theta=\pi} (1-\cos\theta)^2 \sin\theta \, d\theta.$$

$$= \frac{a^2}{2} \left[\frac{(1-\cos\theta)^3}{2}\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{a^2}{6} \left[(1-\cos\pi)^3 - (1-\cos 0)\right] = \frac{4a^2}{3}.$$

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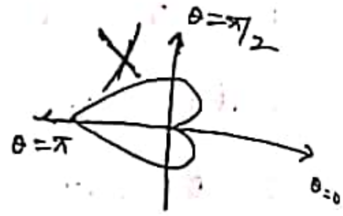
Q) Evaluate $\iint r\sin\theta\,dr\,d\theta$, over the Cardioid $r=a(1+\cos\theta)$ above the initial line.



Sol: Let $I=\iint r\sin\theta\,dr\,d\theta$.

Let $f(r,\theta)=r\sin\theta$.

G.T, the Cardioid $r=a(1+\cos\theta)$

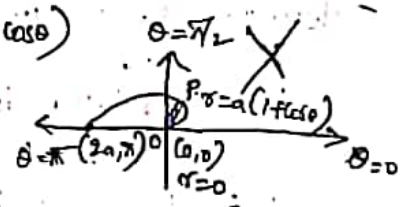
The region of integration above the initial line $\theta=0$.

In this region θ Varies from 0 to π.

$\therefore$ θ limits $\theta=0$, $\theta=\pi$.

Draw a radius vector OP in the region which start at O ($r=0$) and terminates at P ($r=a(1+\cos\theta)$)

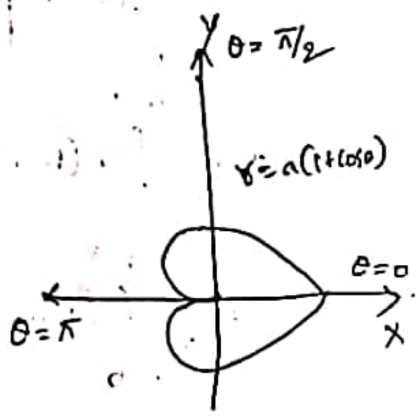
$\therefore$ limits of r is $r=0$, $r=a(1+\cos\theta)$



(The Cardioid is symmetry about initial line. It passes through the origin).

$$I=\iint r\sin\theta\,dr\,d\theta.$$

$$I=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a(1+\cos\theta)} r\sin\theta\,dr\,d\theta.$$



$$=\int_{\theta=0}^{\theta=\pi}\sin\theta\,d\theta\left[\frac{r^2}{2}\right]_{r=0}^{r=a(1+\cos\theta)}$$

$$=\frac{-1}{2}\int_{\theta=0}^{\theta=\pi}a^2(1+\cos\theta)^2(-\sin\theta)\,d\theta.$$

$$=\frac{-1}{2}a^2\left[\frac{(1+\cos\theta)^3}{3}\right]_0^{\pi}$$

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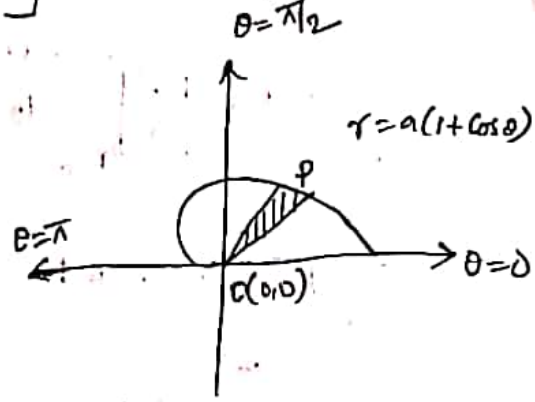
$$= \frac{-a^2}{2}\left[\frac{(1+\cos\pi)^3}{3} - \frac{(1+\cos 0)^3}{3}\right]$$

$$= -\left(\frac{-8a^2}{3\times 2}\right)$$

$$= \frac{8a^2}{6} = \frac{4a^2}{3}$$



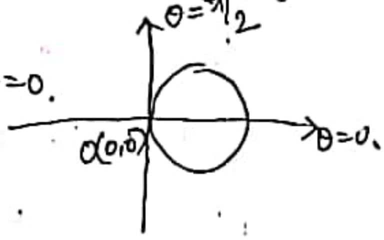
Q) Evaluate $\iint_R r^2 \sin\theta \, dr\, d\theta$, where R is the semicircle of $r = 2a\cos\theta$ above the initial line.

Sol: Let $I = \iint_R r^2 \sin\theta \, dr\, d\theta$.

Let $f(r,\theta) = r^2 \sin\theta$

G.T, The circle $2a\cos\theta$ which passes through the origin center on x-axis.

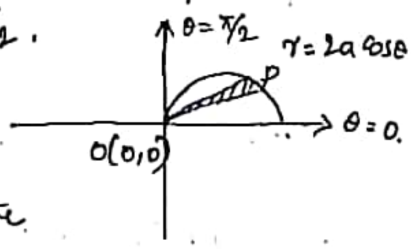
It is symmetry about the initial line $\theta=0$.



It 1

In the region θ varies from 0 to $\pi/2$.

∴ θ limits $\theta=0$, $\theta=\pi/2$



Draw a radius vector OP in the region.

Which start at O $(r=0)$ and terminates at P $(r=2a\cos\theta)$

∴ r limits $r=0$, $r=2a\cos\theta$.

$$I = \iint_R r^2 \sin\theta \, dr\, d\theta.$$

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$$I = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2a\cos\theta} r^2 \sin\theta \, dr\, d\theta.$$

$$= \int_{\theta=0}^{\theta=\pi/2} \sin\theta \, d\theta \left[\frac{r^3}{3}\right]_{r=0}^{r=2a\cos\theta} d\theta$$

$$= \frac{1}{3} \int_{\theta=0}^{\theta=\pi/2} 8a^3 \cos^3\theta \sin\theta \, d\theta$$

$$= \frac{-8a^3}{3} \int_{\theta=0}^{\theta=\pi/2} \cos^3\theta \, (-\sin\theta)\, d\theta$$

$$= -\frac{8a^3}{3} \left[\frac{\cos^4\theta}{4}\right]_{\theta=0}^{\theta=\pi/2} = \frac{2a^3}{3}.$$

Q) Evaluate $\iint r^2 \, dr\, d\theta$ over the area b/w the circles $r = a\sin\theta$ and $r = 2a\sin\theta$ where $a > 0$.

Sol: Let $I = \iint r^2 \, dr\, d\theta$.

Let $f(r,\theta) = r^2$.

G.T, the circles $r = a\sin\theta$, $r = 2a\sin\theta$ which are passes through the origin and centre on y-axis.

In the region θ varies from 0 to π.

$\therefore$ θ limits $\theta = 0$, $\theta = \pi$.

Draw a radius vector OPQ in the region. which enters the region at P ($r = a\sin\theta$) & terminates at Q ($r = 2a\sin\theta$).

$\therefore$ r limits $r = a\sin\theta$, $r = 2a\sin\theta$.

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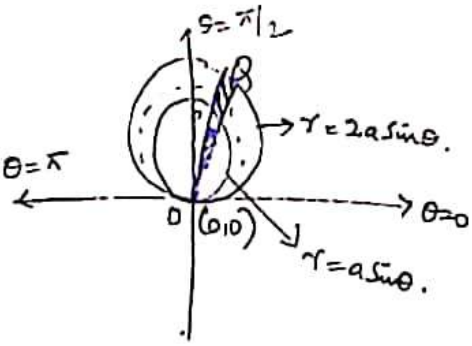

$$I = \iint r^2\, dr\, d\theta.$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=a\sin\theta}^{r=2a\sin\theta} r^2\, dr\, d\theta.$$

$$= \int_{\theta=0}^{\theta=\pi} d\theta \left[\frac{r^3}{3}\right]_{r=a\sin\theta}^{r=2a\sin\theta}$$

$$= \frac{1}{3}\int_{\theta=0}^{\theta=\pi} \left[8a^3\sin^3\theta - a^3\sin^3\theta\right] d\theta$$

$$= \frac{7a^3}{3}\int_{\theta=0}^{\theta=\pi} \sin^3\theta\, d\theta.$$

$$= \frac{7a^3}{3}\int_{\theta=0}^{\theta=\pi} \left[\frac{3\sin\theta - \sin 3\theta}{4}\right] d\theta$$

$$= \frac{7a^3}{12}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_{\theta=0.}^{\theta=\pi}$$

$$= \frac{7a^3}{12}\left[\left\{-3\cos\pi + \frac{\cos 3\pi}{3}\right\} - \left\{-3 + \frac{1}{3}\right\}\right]$$

$$= \frac{7a^3}{12}\left[3 - \frac{1}{3} + 3 - \frac{1}{3}\right] = \frac{7a^3}{12}\left[6 - \frac{2}{3}\right] = \frac{7a^3}{\cancel{12}_{\cancel{6}\,3}}\left[\frac{\cancel{16}^{\cancel{8}\,4}}{3}\right] = \frac{28a^3}{9}$$

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6) Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

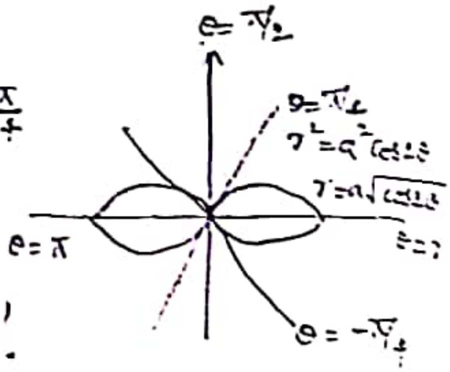
Sol: Let $I = \iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$

let $f(r,\theta) = \frac{r}{\sqrt{a^2+r^2}}$

G.T, the lemniscate $r^2=a^2\cos 2\theta$;

the loop of the lemniscate lies b/w the lines

$\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$



$\therefore$ θ limits are $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$

$\therefore$ Draw a radius vector OP in the region which start at 'O' $(r=0)$ to terminate at 'P'. which lies on lemniscate $(r^2 = a^2\cos 2\theta) \Rightarrow r = a\sqrt{\cos 2\theta}$

$\therefore$ limits $r=0$ & $r = a\sqrt{\cos 2\theta}$

$$I = \int_{\theta=-\pi/4}^{\theta=\pi/4} \int_{r=0}^{r=a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$$= \frac{1}{2} \int_{\theta=-\frac{\pi}{4}}^{\theta=\pi/4} \int_{r=0}^{r=a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$$= \frac{1}{2} \int_{\theta=-\pi/4}^{\theta=\pi/4} \int_{r=0}^{r=a\sqrt{\cos 2\theta}} (a^2+r^2)^{-1/2}(2r)\,dr\,d\theta.$$

$$= \frac{1}{2} \int_{\theta=-\pi/4}^{\theta=\pi/4} \left[\frac{(a^2+r^2)^{1/2}}{\frac{1}{2}} \right]_{r=0}^{r=a\sqrt{\cos 2\theta}} d\theta.$$

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$$= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left[(a^2+a^2\cos 2\theta)^{1/2} - a \right] d\theta$$

$$= a \int_{\theta=-\pi/4}^{\theta=\pi/4} \left[\sqrt{1+\cos 2\theta} - 1 \right] d\theta$$

$$= a \int_{\theta=-\pi/4}^{\theta=\pi/4} \left(\sqrt{2}\cos\theta - 1 \right) d\theta$$

$$= a \left[\sqrt{2}\sin\theta - \theta \right]_{\theta=-\pi/4}^{\theta=\pi/4}$$

$$= a \left[\left(\sqrt{2}\sin\left(\frac{\pi}{4}\right) - \frac{\pi}{4} \right) - \left(\sqrt{2}\sin\left(\frac{-\pi}{4}\right) + \frac{\pi}{4} \right) \right]$$

$$= a \left[\left(1 - \frac{\pi}{4} \right) - \left(-1 + \frac{\pi}{4} \right) \right]$$

$$= a \left(2 - \frac{\pi}{2} \right)$$

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AREA IN POLAR COORDINATES

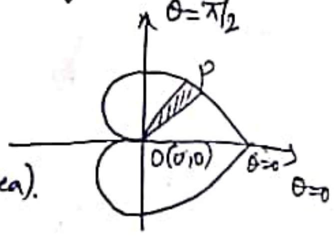
The area of the region bounded by $r=f_1(\theta)$, $r=f_2(\theta)$ & the lines $\theta=\theta_1$ & $\theta=\theta_2$ is given by

$$A = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} r\,dr\,d\theta.$$

Q) Find the area of the Cardiod $r=a(1+\cos\theta)$

Sol: W.K.T the area $A = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} r\,dr\,d\theta.$

G.T, the Cardiod $r=a(1+\cos\theta)$ which is Symmetrical about the initial line $\theta=0$ (x-axis)



In the region θ varies from 0 to 2π (total area).

Draw a radius vector 'OP' in the region which starts at O ($r=0$) & terminates at P $(r=a(1+\cos\theta))$

[$\because$ the Cardiod is in IV Q]

$\therefore$ r limits are $r=0$ & $r=a(1+\cos\theta)$.

$$A = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a(1+\cos\theta)} r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} a^2(1+\cos\theta)^2\,d\theta$$

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6.6

$$= \frac{a^2}{2} \int_{\theta=0}^{\theta=2\pi} \left[1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\theta=2\pi} \left[\frac{3}{2} + \frac{1}{2}\cos 2\theta + 2\cos\theta\right] d\theta$$

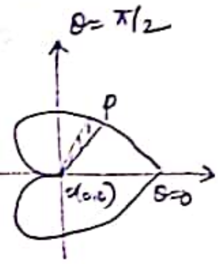
$$= \frac{a^2}{2}\left[\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta + 2\sin\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left[\frac{3}{2}(2\pi) + 0 + 0\right] = \frac{3}{2}a^2\pi$$

Q) Find the area of the Cardiod $r = a(1-\cos\theta)$.

Sol: W.K.T the area $A = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r_2=f_2(\theta)} r\,dr\,d\theta.$

G.T, the Cardiod $r = a(1-\cos\theta)$ which is Symmetric about the initial line $\theta = 0$ (x-axis)



In the region θ varies from 0 to 2π (total area)

Draw a radius vector 'OP' in the region which starts at $O(r=0)$ & terminates at $P(r = a(1-\cos\theta))$

[∵ the Cardiod is in 4Q]

∴ r limits are $r=0$, $r = a(1-\cos\theta)$

$$A = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a(1-\cos\theta)} r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=a(1-\cos\theta)} d\theta$$

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C6

$$= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} a^2 (1-\cos\theta)^2 \, d\theta.$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\theta=2\pi} \left[1 + \left(\frac{1+\cos 2\theta}{2}\right) - 2\cos\theta\right] d\theta.$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\theta=2\pi} \left[\frac{3}{2} + \frac{1}{2}\cos 2\theta - 2\cos\theta\right] d\theta.$$

$$= \frac{a^2}{2} \left[\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta - 2\sin\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{2} \left[\frac{3}{2}(2\pi)\right]$$

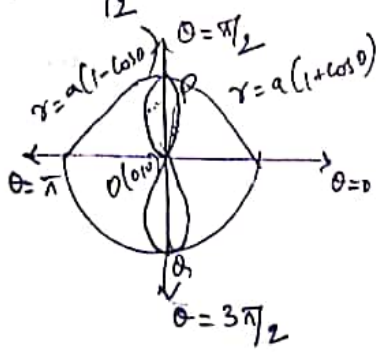
$$= \frac{3a^2\pi}{2} \;//$$

Q) Find the area common to the Cardioid $r=a(1+\cos\theta)$ and $r=a(1-\cos\theta)$.

Sol: G.T, the Cardiods $r=a(1+\cos\theta)$ — ① & $r=a(1-\cos\theta)$ — ②

W.K.T, the area $A = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} r\,dr\,d\theta.$

i) Common area to the Cardiods ① & ② its Symmetric about initial line $\theta=0$ & the line $\theta=\pi/2$



ii) In the first quadrant θ varies from 0 to $\pi/2$

$\therefore$ θ limits $\theta=0$ & $\theta=\pi/2$

iii) Draw a radius vector 'OP' in the region which Start at O $(r=0)$ & terminates at P.

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which lie on $r = a(1-\cos\theta)$ CC

$\therefore$ r limits $r=0$ & $r = a(1-\cos\theta)$.

required area $A = 4$ (Ar of in 1^{st} quad)

$$A = 4\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a(1-\cos\theta)} r\,dr\,d\theta.$$

The point of intersection of ① & ② is

$a(1+\cos\theta) = a(1-\cos\theta)$

$2\cos\theta = 0$

$\theta = \pm\frac{\pi}{2}$

$$A = 4\int_{\theta=0}^{\theta=\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=a(1-\cos\theta)} d\theta.$$

$$= 2\int_{\theta=0}^{\theta=\pi/2} a^2(1-\cos\theta)^2\,d\theta$$

$$= 2a^2\int_{\theta=0}^{\theta=\pi/2}(1+\cos^2\theta - 2\cos\theta)\,d\theta$$

$$= 2a^2\int_{\theta=0}^{\theta=\pi/2}\left(1+\frac{1+\cos2\theta}{2} - 2\cos\theta\right)d\theta.$$

$$= 2a^2\int_{\theta=0}^{\theta=\pi/2}\left[\frac{3}{2}+\frac{1}{2}\cos2\theta - 2\cos\theta\right]d\theta$$

$$= 2a^2\left[\frac{3}{2}\theta + \frac{1}{4}\sin2\theta - 2\sin\theta\right]_0^{\pi/2}$$

$$= 2a^2\left[\frac{3}{2}\left(\frac{\pi}{2}\right)+\frac{1}{4}\sin(2)\left(\frac{\pi}{2}\right) - 2\sin\frac{\pi}{2} - 0\right]$$

$$= 2a^2\left(\frac{3\pi}{4} - 0 - 2\right]$$

$$= 2a^2\left[\frac{3\pi-8}{4}\right]$$

$$= \frac{a^2(3\pi-8)}{2}$$ //

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C6

Q) Find the area b/w the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Sol: Given, the circles $r = 2\sin\theta$, $r = 4\sin\theta$.

The region is symmetric about the line $\theta = \pi/2$

Total area $= 2.$(ar in the 1^{st} Q)

Draw an elementary radius vector OAB from the origin in the region which lies in the 1^{st} quad

OAB enters in the region from the circle $r = 2\sin\theta$ & terminates on at circle $r = 4\sin\theta$.

$\therefore$ r limits are $r = 2\sin\theta$ & $r = 4\sin\theta$

$\therefore$ θ limits are $\theta = 0$ & $\theta = \pi/2$

$$\therefore \text{Area } A = 2\int_{\theta=0}^{\theta=\pi/2}\int_{r=2\sin\theta}^{r=4\sin\theta} r\,dr\,d\theta.$$

$$= 2\int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{r=2\sin\theta}^{r=4\sin\theta}$$

$$= \int_0^{\pi/2} 12\sin^2\theta\,d\theta$$

$$= \int_0^{\pi/2} 6(1-\cos 2\theta)\,d\theta$$

$$= 6\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 6\left(\frac{\pi}{2} - \frac{\sin\pi - \sin 0}{2}\right)$$

$$= 3\pi.$$

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Q) Find the area which lies inside the circle $r = 3a\cos\theta$ & outside the cardiod $r = a(1+\cos\theta)$.

Sol: W.K.T, the area $A = \iint r\,dr\,d\theta$

G.T, the circle $r = 3a\cos\theta$ — ① which passes through the origin and centre on x-axis.

G.T, the cardiod $r = a(1+\cos\theta)$

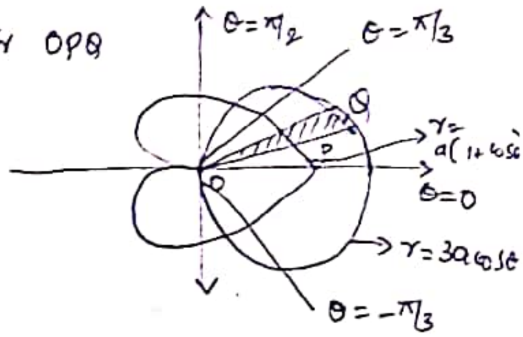
The point of intersection of ① & ② is given by

$$3a\cos\theta = a(1+\cos\theta)$$

$$\boxed{\cos\theta = \tfrac{1}{2}\ \pi/3}$$

The points of intersection of ① & ② is $\theta = -\frac{\pi}{3}$ & $\theta = \pi/3$

Draw an elementary radius vector OPQ which enters the region at $P(r = a(1+\cos\theta))$ & terminates at $Q(r = 3\cos\theta)$.



$\therefore$ r limits $r = a(1+\cos\theta)$ & $r = 3a\cos\theta$.

req Area $A = 2(\text{Area of } 1^{st} Q)$.

$$A = 2\int_{\theta=0}^{\theta=\pi/3}\int_{r=a(1+\cos\theta)}^{r=3a\cos\theta} r\,dr\,d\theta.$$

$$= 2\int_{\theta=0}^{\theta=\pi/3}\left[\frac{r^2}{2}\right]_{r=a(1+\cos\theta)}^{r=3a\cos\theta}$$

$$= \int_{\theta=0}^{\theta=\pi/3}\left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right]d\theta.$$

$$= \int_{\theta=0}^{\theta=\pi/3}\left[8a^2\cos^2\theta - a^2 - 2a^2\cos\theta\right]d\theta.$$

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$$= \int_{\theta=0}^{\theta=\pi/3} 8a^2\left[\frac{1+\cos 2\theta}{2}\right] - a^2 - 2a^2\cos\theta \Big] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/3} \left[3a^2 + 4a^2\cos 2\theta - 2a^2\cos\theta\right] d\theta$$

$$= \left[3a^2\theta + 2a^2\sin 2\theta - 2a^2\sin\theta\right]_{\theta=0}^{\theta=\pi/3}$$

$$= \left[3a^2\left(\frac{\pi}{3}\right) + 2a^2\left(\sin\left(\frac{2\pi}{3}\right)\right) - 2a^2\sin\left(\frac{\pi}{3}\right)\right].$$

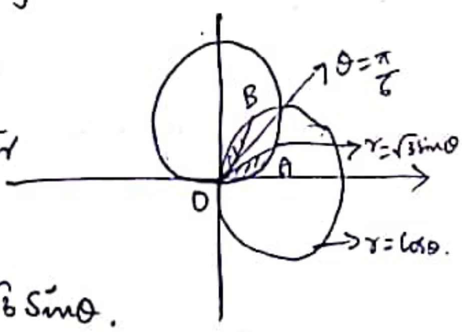
$$= a^2\pi.$$

Q) Find the common area to the circles $r = \cos\theta$ & $r = \sqrt{3}\sin\theta$.

Sol: The point of intersection of the circle $r=\cos\theta$ & $r=\sqrt{3}\sin\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

Hence $\theta = \frac{\pi}{6}$ at P.

Divide the region OAPBO into two $\theta = \frac{\pi}{2}$ sub regions OAP & OBP. Draw an elementary radius vector in each sub region.



i) In subregion OAP, radius vector OA starts from the origin & terminate on the circle $r = \sqrt{3}\sin\theta$.

'r' limits are $r=0$ & $r=\sqrt{3}\sin\theta$.

'θ' limits are $\theta=0$ & $\theta=\pi/6$.

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ii) In the Sub region OBP, the radius Vector OB starts from the region & terminates on the circle $r = \cos\theta$.

CG

'θ' limits are $\theta = \frac{\pi}{6}$ & $\theta = \frac{\pi}{2}$

$$\text{Area } A = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\sqrt{3}\sin\theta} r\,dr\,d\theta + \int_{\theta=\pi/6}^{\theta=\pi/2} \int_{r=0}^{r=\cos\theta} r\,dr\,d\theta.$$

$$= \int_0^{\pi/6} \left[\frac{r^2}{2}\right]_0^{\sqrt{3}\sin\theta} d\theta + \int_{\pi/6}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} 3\sin^2\theta\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \cos^2\theta\,d\theta.$$

$$= \frac{3}{2}\int_0^{\pi/6} \left(\frac{1-\cos 2\theta}{2}\right) d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta.$$

$$= \frac{3}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/6} + \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/2}$$

$$= \frac{3}{4}\left[\frac{\pi}{6} - \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right] + \frac{1}{4}\left[\frac{\pi}{2} - \frac{\pi}{6} + \frac{1}{2}\sin\pi - \frac{1}{2}\sin\frac{\pi}{3}\right]$$

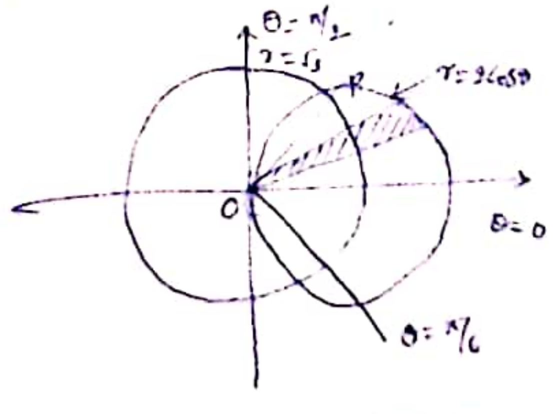
$$= \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$$

Q) Find the area of Cresent bound by the circle $r = \sqrt{3}$ & $r = 2\cos\theta$.

Sol: The points of intersection of $r = \sqrt{3}$ & $r = 2\cos\theta$ are obtained as $\sqrt{3} = 2\cos\theta$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \pm\frac{\pi}{6}.$$



Hence $\theta = \pi/6$ at P.

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The region is symmetric about the initial line $\theta=0$.

Area of the crescent $= 2.$(area above the initial line, $\theta=0$)

Draw an elementary radius vector OAB from the origin in the region above the initial line OAB enters in the region from the circle $r=\sqrt{3}$ & terminates on at the circle $r=2\cos\theta$.

'r' limits are $r=\sqrt{3}$ & $r=2\cos\theta$

'θ' limits are $\theta=0$ & $\theta=\pi/6$.

Area $A = 2\int_{\theta=0}^{\theta=\pi/6}\int_{r=\sqrt{3}}^{r=2\cos\theta} r\,dr\,d\theta.$

$$= 2\int_{\theta=0}^{\theta=\pi/6}\left(\frac{r^2}{2}\right)_{r=\sqrt{3}}^{r=2\cos\theta}$$

$$= \int_{\theta=0}^{\theta=\pi/6}(2^2\cos^2\theta-3)$$

$$= \int_{\theta=0}^{\theta=\pi/6}(4\cos^2\theta-3)$$

$$= \int_{\theta=0}^{\theta=\pi/6}\left(2(1+\cos 2\theta)-3\right)d\theta$$

$$= \left[2\left(\frac{\sin 2\theta}{2}\right)-\theta\right]_0^{\pi/6}$$

$$= \sin\frac{\pi}{3}-\frac{\pi}{6} = \frac{\sqrt{3}}{2}-\frac{\pi}{6}.$$

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Change of Variables in Double Integral

Change of Variables from Cartesian to polar co-ordinates

We have $\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv$.

Here. $x = f(u,v)$ $y = g(u,v)$.

In this case $u=r, v=\theta$.

$x = r\cos\theta$ $y = r\sin\theta$.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$J = r\cos^2\theta + r\sin^2\theta$$

$$J = r \implies |J| = r.$$

Hence eqn (1) becomes $\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

This corresponds to $\iint F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta=\theta_2}\int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r,\theta)\,dr\,d\theta$

To transform cartesian to polar coordinates put $x = r\cos\theta$ $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.

→ Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$. by transforming into polar coordinates.

Sol: Let $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$

Let $f(x,y) = y\sqrt{x^2+y^2}$

In the given Integral, the inner integral limits are in terms of 'x'.

So these are limits of y.

$\therefore$ y limits are $y=0$ & $y = \sqrt{a^2-x^2} \implies x^2+y^2 = a^2$ (circle)

The outer integral limits are constants which are limits of x.

$\therefore$ x limits are $x=0$, $x=a$.

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The region of integration is 1st quadrant of the circle $x^2+y^2=a^2$.
Is bounded by OABO.

To transform into polar co-ordinates we substitute $x=r\cos\theta$ $y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

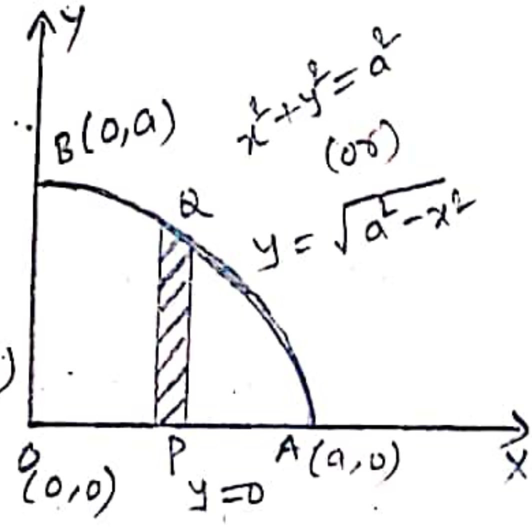
We have $x^2+y^2=a^2 \Rightarrow r=a$

In the region (1st quadrant) θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ θ limits are $\theta=0$, $\theta=\frac{\pi}{2}$.

Draw a radius vector OP in the region which start at O $(r=0)$ and terminates at P $(r=a)$.

$\therefore$ r limits are $r=0$ & $r=a$.



$$I=\int_0^a\int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$$

$$=\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a}(r\sin\theta)\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}\,(r\,dr\,d\theta)$$

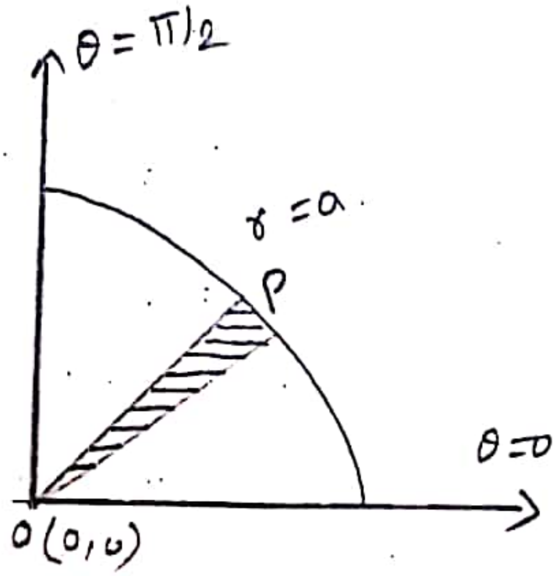
$$=\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a} r^3\sin\theta\,dr\,d\theta$$

$$=\int_{\theta=0}^{\theta=\pi/2}\sin\theta\,d\theta\int_{r=0}^{r=a} r^3\,dr.$$

$$=\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi/2}\left[\frac{r^4}{4}\right]_{r=0}^{r=a}$$

$$=\left[\cos\tfrac{\pi}{2}+\cos 0\right]\left[\frac{a^4}{4}-0\right]$$

$$=\frac{a^4}{4}.$$



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Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx$ by transforming into polar coordinates.

Sol:- Let $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx$.

Let $f(x,y) = \frac{x}{x^2+y^2}$

In the given integral, inner integral limits in terms of x so these are limits of y.

$\therefore$ y limits are $y=0$, $y=\sqrt{2x-x^2}$ $(x^2+y^2-2x=0)$.

The outer integral limits are constants. These are limits of x.

$\therefore$ x limits are $x=0$, $x=2$.

The region of integration bounded by the circle $x^2+y^2-2x=0$. $(y^2=\sqrt{2x-x^2})$ and the lines $x=0$, $y=0$.

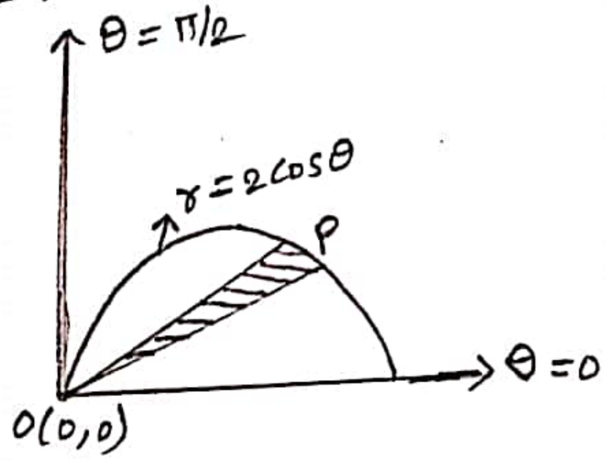
Since the limits of x and y are +ve, the region of integration is the part of the circle in the first quadrant.

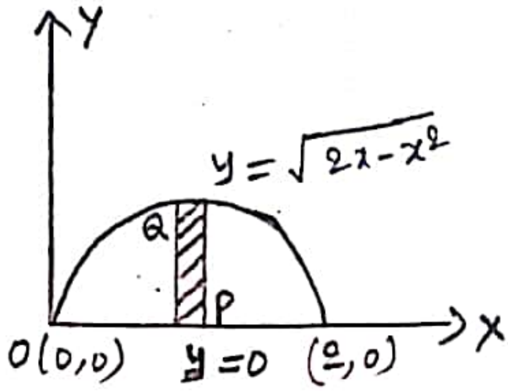
To transform it into polar co ordinates putting $x=r\cos\theta$, $y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

Polar form of the circle $x^2+y^2-2x=0$ is $r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$

$\Rightarrow r = 2\cos\theta$.





In the region θ varies from 0 to $\pi/2$

$\therefore$ θ limits $\theta=0$, $\theta=\frac{\pi}{2}$

Draw an elementary radius vector OP in the region. which starts from origin $(r=0)$ and terminates on the circle at P $(r=2\cos\theta)$

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$\therefore$ r limits $r=0$, $r=2\cos\theta$.

$$\therefore I = \int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2\cos\theta} \frac{r\cos\theta}{r^2(\cos^2\theta+\sin^2\theta)}\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2\cos\theta} \cos\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \cos\theta\,[r]_{r=0}^{r=2\cos\theta}\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} [2\cos\theta-0]\cos\theta\,d\theta$$

$$= 2\int_{\theta=0}^{\theta=\pi/2} \cos^2\theta\,d\theta$$

$$= \cancel{2}\int_{\theta=0}^{\theta=\pi/2} \frac{1+\cos 2\theta}{\cancel{2}}\,d\theta$$

$$= \left[\theta+\frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2}$$

$$= \left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right)-0$$

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dx\,dy = \frac{\pi}{2}.$$

→ Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the region bounded by the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$. $(a>b)$.

Sol:- Let $I = \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$.

Let $f(x,y) = \frac{x^2y^2}{x^2+y^2}$

Given that the circles $x^2+y^2=a^2$ —① $x^2+y^2=b^2$ —②

The region of integration is bounded by ① and ②

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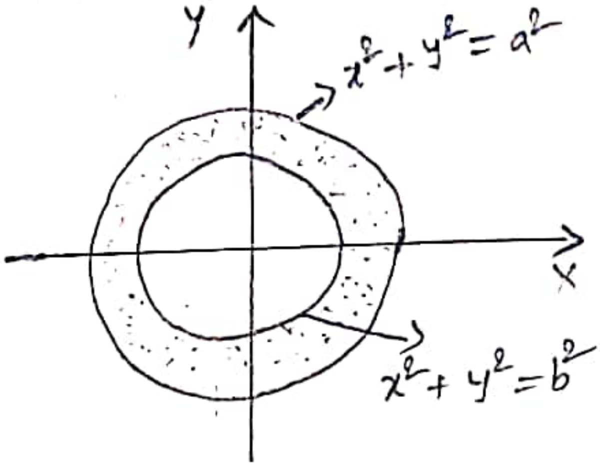
To transform it into polar co ordinates we substitute $x=r\cos\theta,\ y=r\sin\theta$



$dx\,dy = r\,dr\,d\theta.$

We have $x^2+y^2=a^2$

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2$$

$$r^2=a^2 \Rightarrow r=a.$$

$$x^2+y^2=b^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = b^2$$

$$r^2=b^2 \Rightarrow r=b.$$

The region of integration is bounded between the circles $r=a$ and $r=b$.

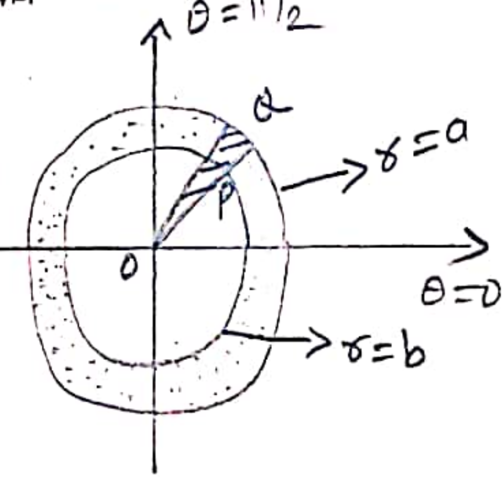
In the region θ varies from 0 to 2π.

$\therefore$ θ limits $\theta=0$ to $\theta=2\pi$.

Draw an elementary radius vector OPQ

which enters into the region at $P\ (r=b)$ and terminates $Q\ (r=a)$.

$\therefore$ r limits $r=b,\ r=a$.



$$I=\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\theta=2\pi}\int_{r=b}^{r=a} \frac{(r\cos\theta)^2(r\sin\theta)^2}{r^2\cos^2\theta+r^2\sin^2\theta}\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=b}^{r=a} r^3(\sin\theta\cos\theta)^2\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=b}^{r=a} r^3\left(\frac{1}{2}\cdot 2\sin\theta\cos\theta\right)^2 dr\,d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\theta=2\pi} \sin^2 2\theta\,d\theta \cdot \int_{r=b}^{r=a} r^3\,dr$$

$$= \frac{1}{4}\int_{\theta=0}^{\theta=2\pi} \frac{1-\cos 4\theta}{2}\,d\theta \left[\frac{r^4}{4}\right]_{r=b}^{r=a}$$

$$= \frac{1}{32}\left[\theta - \frac{\sin 4\theta}{4}\right]_{\theta=0}^{\theta=2\pi}(a^4-b^4)$$

$$= \frac{a^4-b^4}{32}\left[\left(2\pi - \frac{\sin 8\pi}{4}\right)-0\right]$$

$$\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \frac{(a^4-b^4)\pi}{16}$$

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→ Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$ by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Sol:- Let $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$.

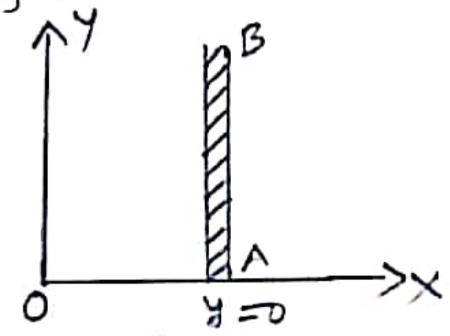
Let $f(x,y) = e^{-(x^2+y^2)}$.

In the given integral all the four limits are constants.

x limits are $x=0,\ x\to\infty$ y limits are $y=0,\ y\to\infty$.

∴ The region of integration is the first quadrant of the xy-plane.

To change it into polar coordinates substituting $x = r\cos\theta,\ y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.

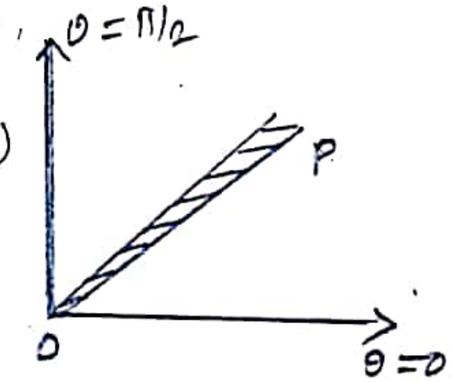


In the region θ varies from 0 to $\pi/2$

∴ θ limits $\theta = 0,\ \theta = \frac{\pi}{2}$.

Draw a radius vector OP in the region, which starts at O ($r=0$) and extends upto ∞

∴ r limits $r=0,\ r\to\infty$



$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r\to\infty} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{r=0}^{r\to\infty} e^{-r^2} r\,dr$$

$$= \left[\theta\right]_{\theta=0}^{\theta=\pi/2} \cdot -\frac{1}{2} \int_{r=0}^{r\to\infty} e^{-r^2} (-2r)\,dr$$

$$= \left(\frac{\pi}{2} - 0\right) \cdot -\frac{1}{2} \cdot \left[e^{-r^2}\right]_{r=0}^{r\to\infty}$$

$$= -\frac{\pi}{4} \left[e^{-\infty} - e^{0}\right]$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}.$$

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We have $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \frac{\pi}{4}$

$$\int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2} dx\, dy = \frac{\pi}{4}$$

$$\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \frac{\pi}{4}$$

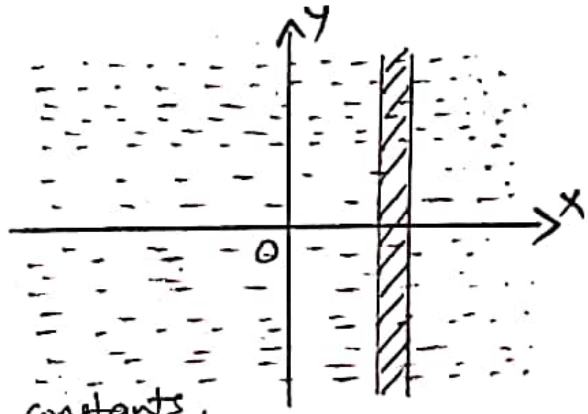
Put $y = x$

$$\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-x^2} dx = \frac{\pi}{4}$$

$$\left[\int_0^\infty e^{-x^2} dx\right]^2 = \frac{\pi}{4}$$

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Evaluate $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\,dy}{(1+x^2+y^2)^{3/2}}$



Sol: Let $I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\, dy}{(1+x^2+y^2)^{3/2}}$

Let $f(x,y) = \frac{1}{(1+x^2+y^2)^{3/2}}$

In the given integral all 4 limits are constants.

Limits of x : $x \to -\infty$ to $x \to \infty$

Limits of y : $y \to -\infty$ to $y \to \infty$.

The region of integration is the entire coordinate plane.

To transform it into polar coordinates put $x = r\cos\theta$ $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

Draw an elementary radius vector which starts from origin and extends upto ∞.

Limits of r : $r = 0$ to $r \to \infty$

Limits of θ : $\theta = 0$ to $\theta = 2\pi$

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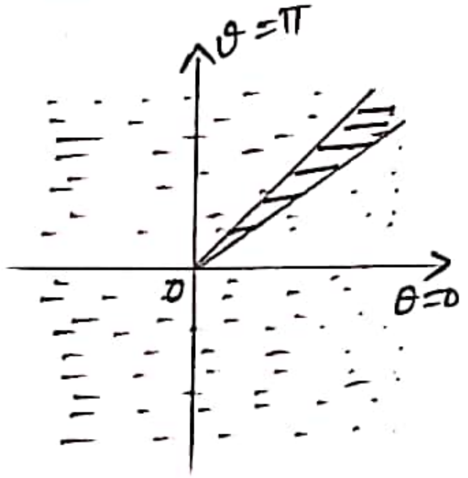
$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(1+x^2+y^2)^{3/2}} = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r\to\infty} \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}}$$



$$= \frac{1}{2}\int_{r=0}^{r\to\infty} (1+r^2)^{-3/2}(2r)\,dr \int_{\theta=0}^{\theta=2\pi} d\theta$$

$$= \frac{1}{2}\left[-2(1+r^2)^{-1/2}\right]_{r=0}^{r\to\infty}\left[\theta\right]_{\theta=0}^{\theta=2\pi}$$

$$= -\left[\frac{1}{(1+r^2)^{1/2}}\right]_{r=0}^{r\to\infty}(2\pi-0).$$

$$= -(0-1)\,2\pi$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(1+x^2+y^2)^{3/2}} = 2\pi.$$

Evaluate $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ by transforming into polar coordinates.

sol: Let $I = \int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

Let $f(x,y) = \frac{x}{x^2+y^2}$.

In the given integral the inner integral limits are interms of y so these are limits of x.

$\therefore$ x limits $x=y$, $x=a$.

The outer integral limits are constants, these are limits of y.

$\therefore$ y limits $y=0$, $y=a$.

The region of Integration is bounded by the lines $y=x$, $x=a$, and $y=0$

The region of Integration is bounded by the lines ~~Putting~~ $\cancel{x=}$. To transform into polar co ordinates $x=r\cos\theta$, $y=r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.

Polar form of (i) the line $y=x$ is $r\sin\theta = r\cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

(ii) the line $x=a$ is $r\cos\theta = a \Rightarrow r = a\sec\theta$.

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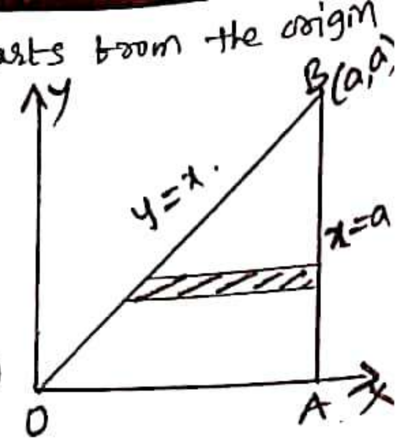
Draw an elementary radius vector OP, which starts from the origin and terminates on the line $r = a\sec\theta$.



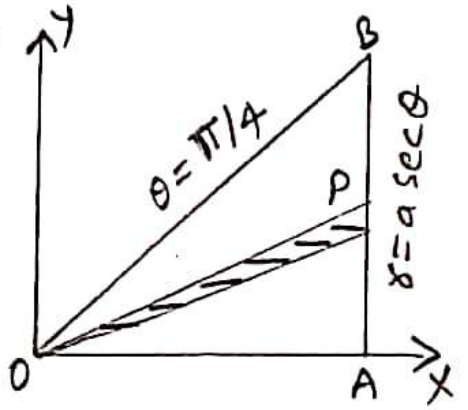
Limits of r : $r=0$ to $r = a\sec\theta$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{4}$.

$$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=a\sec\theta} \frac{r\cos\theta}{r^2}\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=a\sec\theta} \cos\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/4} \cos\theta\, d\theta\, \left[r\right]_{r=0}^{r=a\sec\theta}$$



$$= \int_{\theta=0}^{\theta=\pi/4} a\cos\theta\sec\theta\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/4} a\, d\theta$$

$$= \left[a\theta\right]_{\theta=0}^{\theta=\pi/4}$$

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \frac{a\pi}{4}$$

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Change of variables from polar to cartesian :-

→ Transform the following to cartesian form and hence evaluate.

$\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr \, d\theta$.

sol: Let $I = \int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr\, d\theta$

$$I = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a} (r\sin\theta)(r\cos\theta)\, r\,dr\,d\theta \; ; \; f(r,\theta) = r^2 \sin\theta\cos\theta .$$

In the given integral all four limits are constants.

θ limits $\theta=0, \theta=\pi$ $\quad$ r limits $r=0$ $r=a$.

In cartesian coordinates the same region is given by

$x=0, y=0$ ($\because r=0$) and $x^2+y^2=a^2$ ($\because r=a$)

Since θ varies from 0 to π, the region of integration is the semi-circle $x^2+y^2=a^2$.

Changing to cartesian coordinates substitute $r\cos\theta = x$, $r\sin\theta = y$

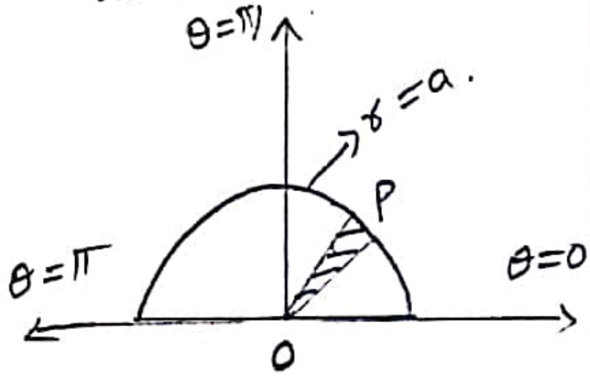
$r\,dr\,d\theta = dx\,dy$.

→ Draw a vertical strip PQ in the region

We have to fix x first.

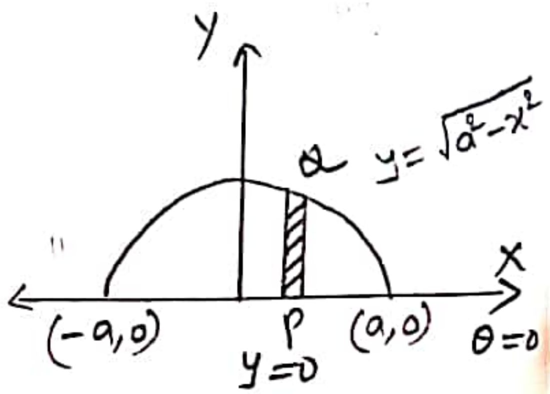
In the region x varies from $-a$ to a

$\therefore$ x limits $x=-a$, $x=a$.



For each x, y varies from a point P on the x-axis ($y=0$) to a point Q on the circle $x^2+y^2=a^2$ i.e $y=\sqrt{a^2-x^2}$

$\therefore$ y limits $y=0$, $y=\sqrt{a^2-x^2}$



$$I = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a} (r\sin\theta)(r\cos\theta)\, r\,dr\,d\theta$$

$$= \int_{x=-a}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} xy \, dy\, dx .$$

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$$= \int_{x=-a}^{x=a} x \cdot \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_{x=-a}^{x=a} x\,(a^2-x^2)\, dx$$

$$= \frac{1}{2} \int_{x=-a}^{x=a} (a^2x - x^3)\, dx$$

$$= \frac{1}{2} \left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{x=-a}^{x=a}$$

$$= \frac{1}{2} \left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - \left(\frac{a^4}{2} - \frac{a^4}{4}\right)\right]$$

$$\int_0^{\pi}\int_0^{r} r^3 \sin\theta \cos\theta \, dr\, d\theta = 0$$

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Triple Integrals :—

Let $f(x,y,z)$ be a function defined over a 3-dimensional finite region V.
Divide the region V into n elementary volumes (sub-divisions) $\delta v_1, \delta v_2,$
$\cdots \delta v_n$. Let (x_i, y_i, z_i) be any point in the ith sub division δv_i.

Consider the sum $\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\delta v_i$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta v_i \to 0$ is called the
triple integral of $f(x,y,z)$ over the region V and is denoted by $\iiint f(x,y,z)\,dv$.

Evaluation of Triple Integral :—

The triple integral is evaluated as the repeated integral.

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$$

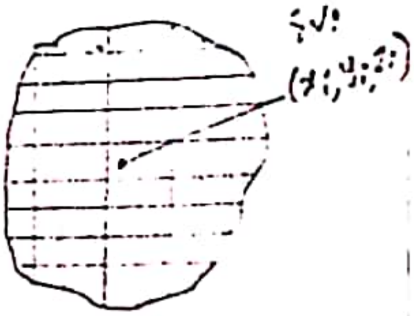
Where the limits of z are z_1, z_2 which are either constants or functions of x and y; The y limits y_1, y_2 are either constants or function of x; The x limits x_1, x_2 are constants.

The above multiple integral is evaluated as follows.
First $f(x,y,z)$ is integrated w.r.t z between the limits z_1 and z_2, keeping x and y are fixed. The resulting expression is integrated w.r.t y between the limits y_1 and y_2 keeping x constant. The result is finally integrated w.r.t x from x_1 to x_2.

Thus the above multiple integral can be evaluated as.

$$\int_{x=x_1}^{x=x_2}\left[\int_{y=y_1(x)}^{y=y_2(x)}\left[\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx$$

<u>Note</u>:- When x_1, x_2, y_1, y_2 and $z_1\ z_2$ are all constants, the triple integral can be evaluated in any order.



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→ Evaluate $\int_0^1 \int_1^2 \int_2^3 xyz\, dz\, dy\, dx$.

Sol: Let $I = \int_0^1 \int_1^2 \int_2^3 xyz\, dz\, dy\, dx$.

Here all the limits of x, y and z are constants so we can integrate in any order.

$$I = \int_0^1 \int_1^2 \int_2^3 xyz\, dz\, dy\, dx = \int_{x=0}^{x=1} \int_{y=1}^{y=2} \int_{z=2}^{z=3} xyz\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1} x\, dx \int_{y=1}^{y=2} y\, dy \int_{z=2}^{z=3} z\, dz$$

$$= \left[\frac{x^2}{2}\right]_{x=0}^{x=1} \left[\frac{y^2}{2}\right]_{y=1}^{y=2} \left[\frac{z^2}{2}\right]_{z=2}^{z=3}$$

$$= \left(\frac{1}{2} - 0\right)\left(\frac{4}{2} - \frac{1}{2}\right)\left(\frac{9}{2} - \frac{4}{2}\right)$$

$$= \frac{15}{8}.$$

→ Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$

Sol: Let $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$.

In the given integral, inner integral limits are interms of x and y so these are limits of z. Second integral limits are interms of x so these are limits of y. and outer integral limits are constants these are limits of x.

$$I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$$

$$I = \int_{x=0}^{x=a} \left[\int_{y=0}^{y=x} \left[\int_{z=0}^{z=x+y} e^{x+y} \cdot e^{z}\, dz \right] dy \right] dx$$

$$= \int_{x=0}^{x=a} \int_{y=0}^{y=x} e^{x+y} \cdot \left[e^{z}\right]_{z=0}^{z=x+y} dy\, dx.$$

$$= \int_{x=0}^{x=a} \int_{y=0}^{y=x} e^{x+y} \left[e^{x+y} - e^{0}\right] dy\, dx.$$

$$= \int_{x=0}^{x=a} \left[\int_{y=0}^{y=x} \left[e^{2x} \cdot e^{2y} - e^{x} \cdot e^{y}\right] dy \right] dx$$

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$$I = \int_{x=0}^{x=a} \left[\frac{e^{2x}.e^{2y}}{2} - e^{x}.e^{y}\right]_{y=0}^{y=x} dx$$

$$= \int_{x=0}^{x=a} \left\{\left[\frac{e^{2x+2x}}{2} - e^{2x}\right] - \left[\frac{e^{2x}}{2} - e^{x}\right]\right\} dx$$

$$= \int_{x=0}^{x=a} \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right] dx$$

$$= \int_{x=0}^{x=a} \left[\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^{x}\right] dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x}\right]_{x=0}^{x=a}$$

$$= \left[\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a}\right] - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{1}{8}.$$

→ Evaluate $\int_{1}^{e}\int_{1}^{\log y}\int_{1}^{e^x} \log z \, dz \, dx \, dy$

Sol: Let $I = \int_{1}^{e}\int_{1}^{\log y}\int_{1}^{e^x} \log z \, dz \, dx \, dy$

$$I = \int_{y=1}^{y=e}\left[\int_{x=1}^{x=\log y}\left[\int_{z=1}^{z=e^x} \log z \, dz\right] dx\right] dy$$

$$= \int_{y=1}^{y=e}\int_{x=1}^{x=\log y} \left[z\log z - z\right]_{z=1}^{z=e^x} dx.dy$$

$$= \int_{y=1}^{y=e}\int_{x=1}^{x=\log y} \left\{\left[e^{x}\log_e e^{x} - e^{x}\right] - \left[0-1\right]\right\} dx \, dy$$

$$= \int_{y=1}^{y=e}\left[\int_{x=1}^{x=\log y} \left[xe^{x} - e^{x} + 1\right] dx\right] dy$$

$$= \int_{y=1}^{y=e} \left[xe^{x} - e^{x} - e^{x} + x\right]_{x=1}^{x=\log y} dy = \int_{y=1}^{y=e} \left[xe^{x} - 2e^{x} + x\right]_{x=1}^{x=\log y} dy$$

$$= \int_{y=1}^{y=e} \left[y\log y - 2y + \log y + e - 1\right] dy$$

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$$= \left[\frac{y^2}{2}\log y - \frac{y^2}{4} + y\log y - y - y^2 + (e-1)y\right]_{y=1}^{y=e}$$

$$= \left[\frac{e^2}{2}\log_e e - \frac{e^2}{4} + e\log e - e - e^2 + (e-1)e\right] - \left[-\frac{1}{4} - 1 - 1 + (e-1)\right]$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

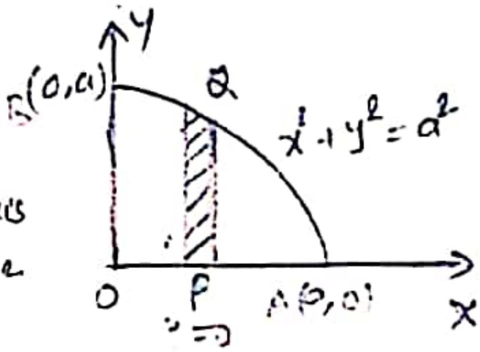
$$= \frac{1}{4}(e^2 - 8e + 13)$$

→ Evaluate $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$

sol: Let $I = \iiint xyz\,dx\,dy\,dz$.

Given sphere is $x^2+y^2+z^2=a^2 \Rightarrow z = \sqrt{a^2-x^2-y^2}$.

The projection of the sphere on the xy-plane is the circle $x^2+y^2=a^2$.



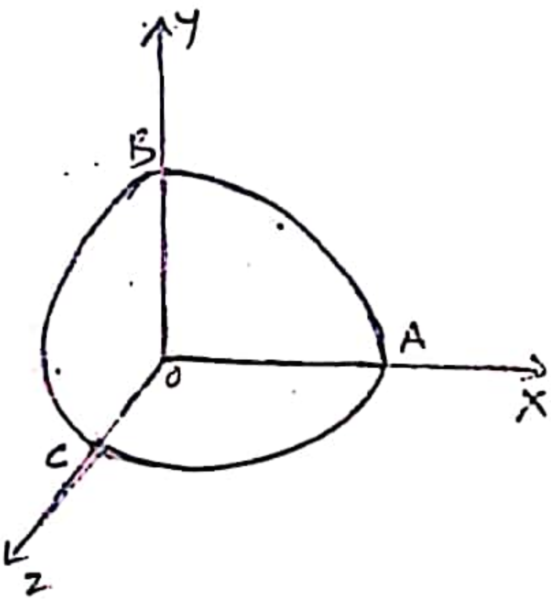
Here x varies from 0 to a.

$\therefore$ x limits $x=0$, $x=a$.

For each x, y varies from a point P on x-axis ($y=0$) to a point Q on the circle $x^2+y^2=a^2$

i.e $y = \sqrt{a^2-x^2}$

$\therefore$ y limits $y=0$, $y=\sqrt{a^2-x^2}$



$I = \iiint xyz\,dx\,dy\,dz$

$$= \int_{x=0}^{x=a}\left[\int_{y=0}^{y=\sqrt{a^2-x^2}}\left[\int_{z=0}^{z=\sqrt{a^2-x^2-y^2}} xyz\,dz\right]dy\right]dx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} xy\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} xy(a^2-x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} x(a^2y - x^2y - y^3)\,dy.dx$$

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$$I = \frac{1}{2}\int_{x=0}^{x=a} x\left[a^2\frac{y^2}{2} - x^2\frac{y^2}{2} - \frac{y^4}{4}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{x=a} x\left[\frac{a^2(a^2-x^2)}{2} - \frac{x^2(a^2-x^2)}{2} - \frac{(a^2-x^2)^2}{4}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{x=a} x\left[\frac{(a^2-x^2)}{2}(a^2-x^2) - \frac{(a^2-x^2)^2}{4}\right] dx$$

$$= \frac{1}{8}\int_{x=0}^{x=a} x(a^2-x^2)^2\, dx.$$

$$= \frac{1}{8}\int_{x=0}^{x=a} (a^4+x^4-2a^2x^2)\, x\, dx.$$

$$= \frac{1}{8}\int_{x=0}^{x=a} (a^4.x + x^5 - 2a^2x^3)\, dx$$

$$= \frac{1}{8}\left[a^4.\frac{x^2}{2} + \frac{x^6}{6} - 2a^2\frac{x^4}{4}\right]_{x=0}^{x=a}$$

$$= \frac{1}{8}\left[\frac{a^6}{2} + \frac{a^6}{6} - \frac{a^6}{2}\right]$$

$$= \frac{1}{8}\cdot\frac{a^6}{6}$$

$$= \frac{a^6}{48}.$$

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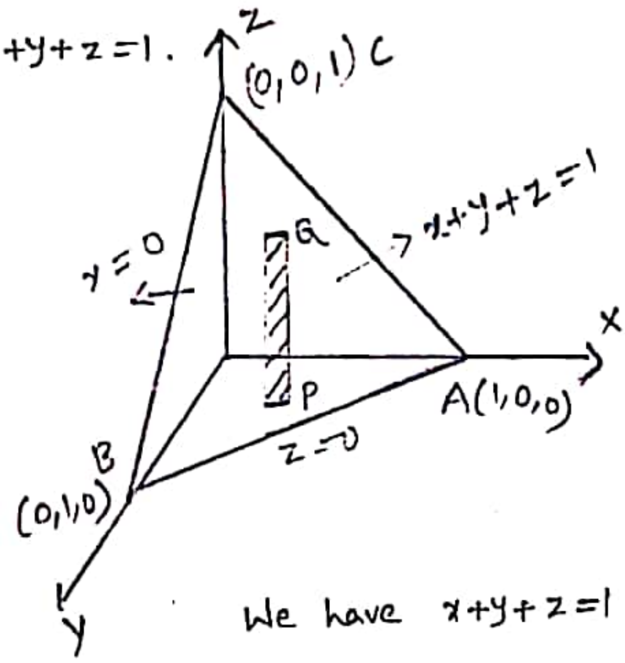
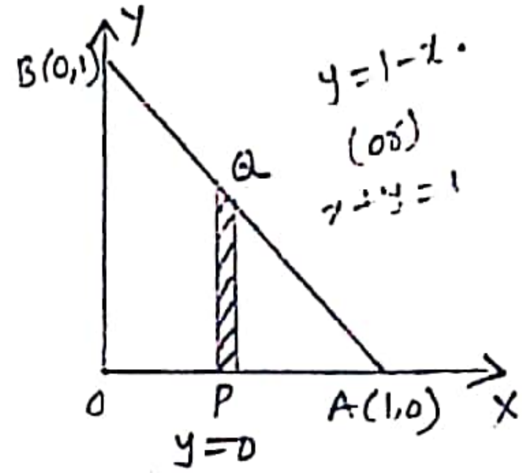
→ Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ taken over the volume bounded by the planes $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$. Ans:- $\frac{1}{16}[8\log 2 - 5]$

Sol:- Let $I = \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$

Let $f(x,y,z) = \frac{1}{(x+y+z+1)^3}$

Given that volume bounded by the planes $x=0, y=0, z=0$ and the plane $x+y+z=1$.





We have $x+y+z=1 \Rightarrow z = 1-x-y$

$\therefore$ z limits $z=0$, $z=1-x-y$

The projection of the region in xy-plane is $\triangle OAB$.

In the region x varies from 0 to 1.

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $x+y=1$ i.e $y=1-x$.

$\therefore$ x limits $x=0, x=1$, y limits $y=0, y=1-x$.

$$I = \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_{z=0}^{z=1-x-y} dy\,dx$$

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$$= \frac{-1}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left\{[x+y+1-x-y+1]^{-2} - [x+y+1]^{-2}\right\}dy\,dx$$

$$= \frac{-1}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left[2^{-2} - [x+y+1]^{-2}\right]dy\,dx$$

$$= \frac{-1}{2}\int_{x=0}^{x=1}\left[\frac{y}{4} - \frac{(x+y+1)^{-2+1}}{-2+1}\right]_{y=0}^{y=1-x} dx$$

$$= \frac{-1}{2}\int_{x=0}^{x=1}\left\{\left[\frac{1-x}{4} - \frac{(x+1-x+1)^{-1}}{-1}\right] - \left[0 - \frac{(x+1)^{-1}}{-1}\right]\right\}dx$$

$$= \frac{-1}{2}\int_{x=0}^{x=1}\left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right]dx$$

$$= \frac{-1}{2}\int_{x=0}^{x=1}\left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right]dx$$

$$= \frac{-1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log|x+1|\right]_{x=0}^{x=1}$$

$$= \frac{-1}{2}\left[\left(\frac{3}{4} - \frac{1}{8} - \log 2\right) - \left(0 - \log 1 + 0\right)\right]$$

$$= \frac{-1}{2}\left(\frac{5}{8} - \log 2\right)$$

$$= \frac{1}{16}(8\log 2 - 5)\ .$$

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→ Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ taken over the volume bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$ and $z=0$, $z=1$.

sol: Let $I = \iiint (x+y+z)\,dx\,dy\,dz$ . . Let $f(x,y,z) = x+y+z$

Given that the volume bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

$$I = \iiint (x+y+z)\,dx\,dy\,dz = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} (x+y+z)\,dz\,dy\,dx.$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} \left[(x+y)z + \frac{z^2}{2}\right]_{z=0}^{z=1} dy\,dx.$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} \left[x+y+\frac{1}{2}\right] dy\,dx$$

$$= \int_{x=0}^{x=1} \left[\left(x+\frac{1}{2}\right)y + \frac{y^2}{2}\right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=1} \left(x+\frac{1}{2}+\frac{1}{2}\right) dx$$

$$= \int_{x=0}^{x=1} (x+1)\,dx.$$

$$= \left[\frac{x^2}{2}+x\right]_{x=0}^{x=1}$$

$$= \frac{1}{2}+1$$

$$= \frac{3}{2}.$$

→ Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the co ordinate planes and the plane $x+y+z=1$.

sol:- Let $I = \iiint (x+y+z)\,dx\,dy\,dz$.

Given that the tetrahedron bounded by the co ordinate planes $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$.

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We have $x+y+z=1 \Rightarrow z=1-x-y$

$\therefore$ z limits $z=0, z=1-x-y$.

The projection of the tetrahedron in xy-plane is ΔOAB.

Draw a vertical strip PQ in the region.

We have to fix x first.

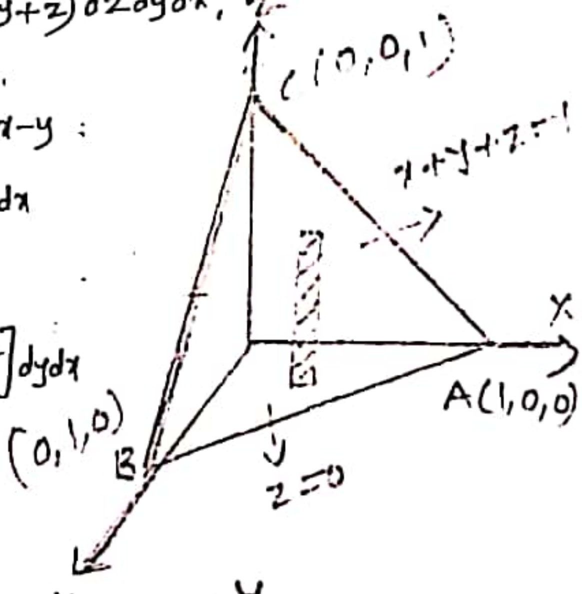
x varies from 0 to 1.

$\therefore$ x limits $x=0, x=1$.

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $x+y=1$ i.e $y=1-x$

$\therefore$ y limits $y=0, y=1-x$.

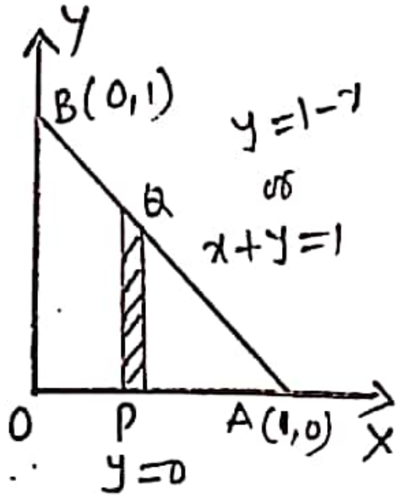
$$I=\iiint_V (x+y+z)\,dx\,dy\,dz=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y}(x+y+z)\,dz\,dy\,dx,$$



$$=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left[\frac{(x+y+z)^2}{2}\right]_{z=0}^{z=1-x-y}dy\,dx$$

$$=\frac{1}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left[(x+y+1-x-y)^2-(x+y)^2\right]dy\,dx$$

$$=\frac{1}{2}\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left[1-(x+y)^2\right]dy\,dx$$



$$=\frac{1}{2}\int_{x=0}^{x=1}\left[y-\frac{(x+y)^3}{3}\right]_{y=0}^{y=1-x}dx.$$

$$=\frac{1}{2}\int_{x=0}^{x=1}\left[(1-x)-\frac{(x+1-x)^3}{3}+\frac{x^3}{3}\right]dx$$

$$=\frac{1}{2}\int_{x=0}^{x=1}\left[1-x-\frac{1}{3}+\frac{x^3}{3}\right]dx.$$

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$$= \frac{1}{2}\int_{x=0}^{x=1}\left[\frac{2}{3} - x - \frac{x^3}{3}\right]dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} - \frac{x^4}{12}\right]_{x=0}^{x=1}$$

$$= \frac{1}{2}\left[\left(\frac{2}{3} - \frac{1}{2} - \frac{1}{12}\right) - 0\right]$$

$$= \frac{1}{2}\cdot\frac{7}{12}$$

$$\iiint (x+y+z)\,dx\,dy\,dz = \frac{7}{24}$$

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Volume as a triple integral :-

Suppose a three dimensional solid is cut into elementary rectangular parallelopiped by drawing planes parallel to the coordinate planes. The volume of an elementary parallelopiped δv is $\delta x \delta y \delta z$

Hence the total volume of the solid is $\iiint_V dv = \iiint_V dx\,dy\,dz$ where the integration is carried over the entire volume.

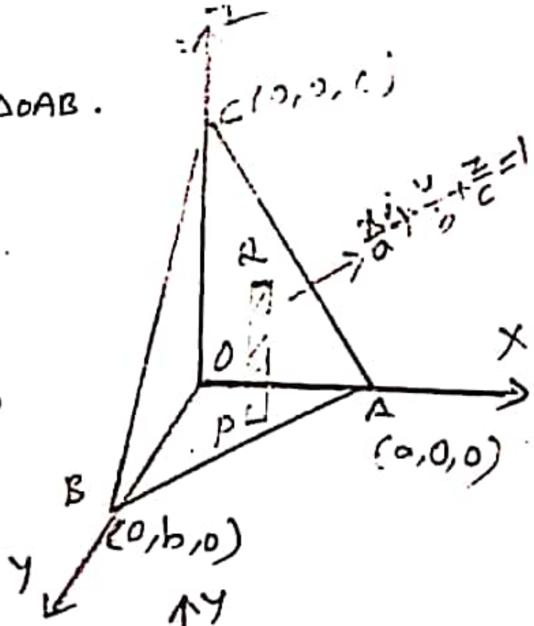
→ Find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol:- The required volume $v = \iiint_V dx\,dy\,dz$..

Given that the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

On the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$\therefore$ z limits $z=0$, $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$.



The projection of tetrahedron in xy-plane is ΔOAB.

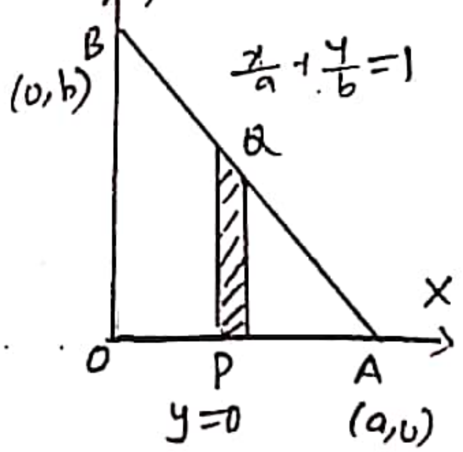
Draw a vertical strip PQ in the region..

We have to fix x first

In the region x varies from 0 to a.

For each x, y ~~varies~~ varies from a point P on x-axis $(y=0)$ to a point Q on the line.

$\frac{x}{a}+\frac{y}{b}=1$ i.e $y=b\left(1-\frac{x}{a}\right)$



$\therefore$ x limits $x=0, x=a$

y limits $y=0$, $y=b\left(1-\frac{x}{a}\right)$

$$\therefore V = \int_{x=0}^{x=a}\int_{y=0}^{y=b\left(1-\frac{x}{a}\right)}\int_{z=0}^{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx.$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b\left(1-\frac{x}{a}\right)} \left[z\right]_{z=0}^{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx.$$

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$$V = \int_{x=0}^{x=a} \int_{y=0}^{y=b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx.$$

$$= \int_{x=0}^{x=a} c\left[\left(1-\frac{x}{a}\right)y - \frac{y^2}{2b}\right]_{y=0}^{y=b(1-\frac{x}{a})} dx.$$

$$= \int_{x=0}^{x=a} c\left[b\left(1-\frac{x}{a}\right)^2 - \frac{b}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2} \int_{x=0}^{x=a} \left(1-\frac{x}{a}\right)^2 dx.$$

$$= \frac{bc}{2} \left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_{x=0}^{x=a}$$

$$= \frac{bc}{2}\left[0 - \left(\frac{-a}{3}\right)\right]$$

$$= \frac{abc}{6}.$$

$\therefore$ Volume of the tetrahedron $= \frac{abc}{6}.$

→ Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol:- Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

The solid figure $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is cut into 8 pieces by the three coordinate planes. Hence the volume of the solid is equal to 8 times the volume of the solid bounded by $x=0$, $y=0$, $z=0$ and the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

For a fixed (x,y) on the xy-plane, z varies from $z=0$ to $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

consider the quadrant of the ellipse in the first quadrant of the xy-plane

For a fixed x, y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

Then x varies from 0 to a.

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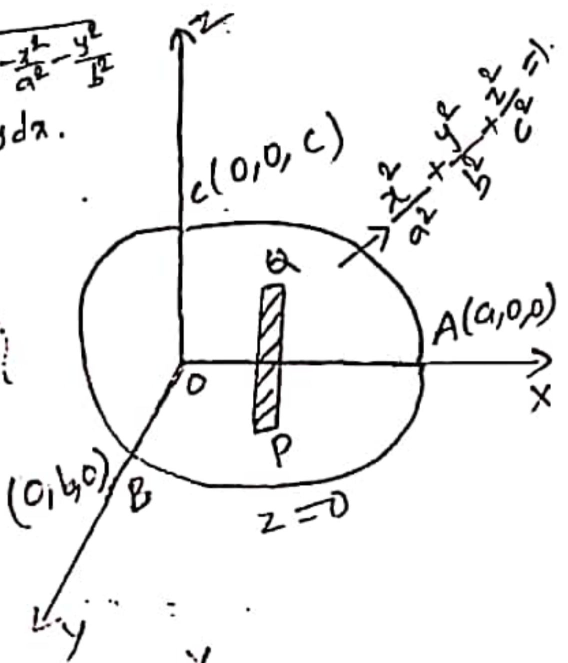
Volume $v = \iiint dx\,dy\,dz$

Required volume $v = 8 \int_{x=0}^{x=a} \int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} \int_{z=0}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx.$

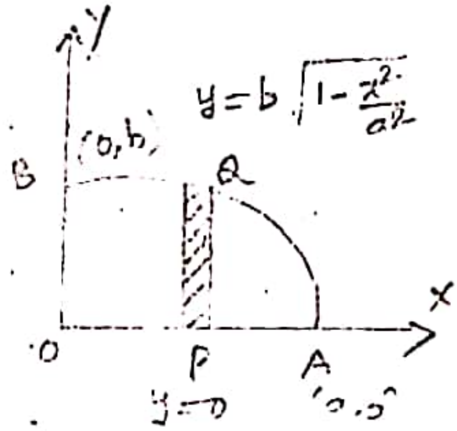


$$v = 8 \int_{x=0}^{x=a} \int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} [z]_{z=0}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$v = 8 \int_{x=0}^{x=a} \int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\,dx$$

Write. $1-\frac{x^2}{a^2} = \frac{p^2}{b^2}$

$$v = 8 \int_{x=0}^{x=a} \int_{y=0}^{y=p} \frac{c}{b}\sqrt{p^2-y^2}\, dy\,dx.$$



$$v = \frac{8c}{b} \int_{x=0}^{x=a} \left[\int_{y=0}^{y=p} \sqrt{p^2-y^2}\, dy\right] dx$$

Put $y = p\sin\theta$

$dy = p\cos\theta\, d\theta$

if $y=0$, $\theta=0$ and if $y=p$, $\theta=\frac{\pi}{2}$.

$$v = \frac{8c}{b} \int_{x=0}^{x=a} \left[\int_{\theta=0}^{\theta=\pi/2} p\cos\theta\; p\cos\theta\, d\theta\right] dx$$

$$= \frac{8c}{b} \int_{x=0}^{x=a} \left[\int_{\theta=0}^{\theta=\pi/2} p^2\cos^2\theta\, d\theta\right] dx$$

$$= \frac{8c}{b} \cdot \int_{x=0}^{x=a} p^2\left[\int_{\theta=0}^{\theta=\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta\right] dx$$

$$= \frac{8c}{b} \int_{x=0}^{x=a} p^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\theta=0}^{\theta=\pi/2} dx.$$

$$= \frac{8c}{b} \int_{x=0}^{x=a} p^2 \left[\left(\frac{\pi}{4} + \frac{\sin\pi}{4}\right) - 0\right] dx$$

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$$= \frac{8c}{b} \int_{x=0}^{x=a} b^2\left(1-\frac{x^2}{a^2}\right)\cdot\frac{\pi}{4}\,dx$$

$$= \frac{2c\pi}{b} b^2 \int_{x=0}^{x=a} \left(1-\frac{x^2}{a^2}\right) dx$$

$$= 2bc\pi \left[x - \frac{x^3}{3a^2}\right]_{x=0}^{x=a}$$

$$= 2bc\pi \left[\left(a - \frac{a^3}{3a^2}\right) - 0\right]$$

Volume $V = \frac{4\pi}{3} abc$

→ Using triple integration, find the volume of the sphere $x^2+y^2+z^2=a^2$

Sol: The solid figure $x^2+y^2+z^2=a^2$ is cut into 8 equal pieces by the three coordinate planes. Hence the volume of the solid is equal to 8 times the volume of the solid bounded by $x=0$, $y=0$, $z=0$ and the surface $x^2+y^2+z^2=a^2$.

For a fixed (x,y) on the xy-plane, z varies from $z=0$ to $z=\sqrt{a^2-x^2-y^2}$.

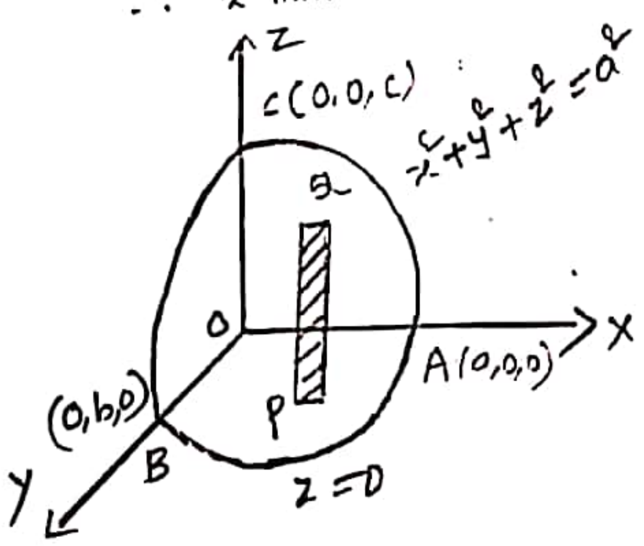
$\therefore$ z limits $z=0$, $z=\sqrt{a^2-x^2-y^2}$

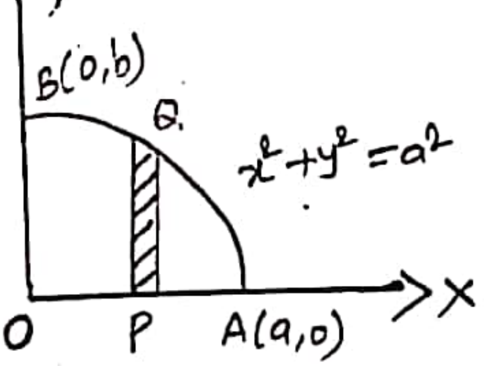
Consider the quadrant of the ~~ellipse~~ circle in the first quadrant of the xy-plane.

For a fixed x, y varies from 0 to $\sqrt{a^2-x^2}$.

Then x varies from 0 to a.

$\therefore$ x limits $x=0$, $x=a$, y limits $y=0$, $y=\sqrt{a^2-x^2}$





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Volume $v = \iiint dx\,dy\,dz$

Required volume $v = 8 \times \iiint dx\,dy\,dz$

$$v = 8 \int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} \int_{z=0}^{z=\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

$$v = 8 \int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} \left[z\right]_{z=0}^{z=\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8 \int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

We write $a^2-x^2 = p^2$

$$= 8 \int_{x=0}^{x=a} \int_{y=0}^{y=p} \sqrt{p^2-y^2}\,dy\,dx.$$

Put $y = p\sin\theta$

$dy = p\cos\theta\,d\theta$

when $y=0$, $\theta=0$

when $y=p$ $\theta=\pi/2$

$$= 8 \int_{x=0}^{x=a} \int_{\theta=0}^{\theta=\pi/2} p\cos\theta\, p\cos\theta\,d\theta\,dx.$$

$$= 8 \int_{x=0}^{x=a} \int_{\theta=0}^{\theta=\pi/2} p^2\cos^2\theta\,d\theta\,dx$$

$$= 8 \int_{x=0}^{x=a} \int_{\theta=0}^{\theta=\pi/2} p^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dx$$

$$= 4 \int_{x=0}^{x=a} p^2 \int_{\theta=0}^{\theta=\pi/2} (1+\cos 2\theta)\,d\theta \cdot dx$$

$$= 4 \int_{x=0}^{x=a} p^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} dx.$$

$$= 4 \int_{x=0}^{x=a} p^2 \left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - 0\right] dx.$$

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$$V = \frac{4\pi}{2}\int_{x=0}^{x=a}(a^2-x^2)\,dx$$

$$= \frac{4\pi}{2}\int_{x=0}^{x=a} a^2-x^2)\,dx$$

$$= \frac{4\pi}{2}\left[a^2x - \frac{x^3}{3}\right]_{x=0}^{x=a}$$

$$= \frac{4\pi}{2}\left[\left(a^3 - \frac{a^3}{3}\right) - 0\right]$$

$$V = \frac{4\pi}{3}a^3.$$

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Change of variables in a triple integral :-

Let the functions $x = \phi_1(u,v,w)$ $y = \phi_2(u,v,w)$ and $z = \phi_3(u,v,w)$ be the transformations from cartesian coordinates to the curvilinear coordinates u, v, w.

The Jacobian for this transformation is given by

Then $\iiint_V f(x,y,z)\, dx\, dy\, dz = \iiint_{V'} f(\phi_1, \phi_2, \phi_3) |J|\, du\, dv\, dw$.

Where v' is the corresponding domain in the curvilinear coordinates u, v, w.

(a) Change of variables from Cartesian to Spherical Polar Coordinate System :-

In problems having symmetry with respect to a point O (generally the origin) it would be convenient to use spherical coordinates with this point chosen as origin.

The relations between the cartesian coordinates x, y, z and spherical polar coordinates r, θ, ϕ. (i.e $u = r$, $v = \theta$, $w = \phi$) are given by :

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

and $dx\,dy\,dz = |J|\, dr\, d\theta\, d\phi$.

Where $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= r^2\sin\theta$

$\therefore \iiint_V f(x,y,z)\, dx\, dy\, dz = \iiint_{V'} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi.$

The region v in (x,y,z) is to be covered by the limits of r, θ, ϕ and is denoted as v'

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(b) change of variables from Cartesian to cylindrical coordinate system :-

The relations between the cartesian coordinates x, y, z and the cylindrical coordinates r, θ, z i.e $u = r$, $v = \theta$, $w = z$ are given by .

$x = r\cos\theta$, $y = r\sin\theta$; $z = z$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\therefore J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$

Limits of Spherical coordinate System :-

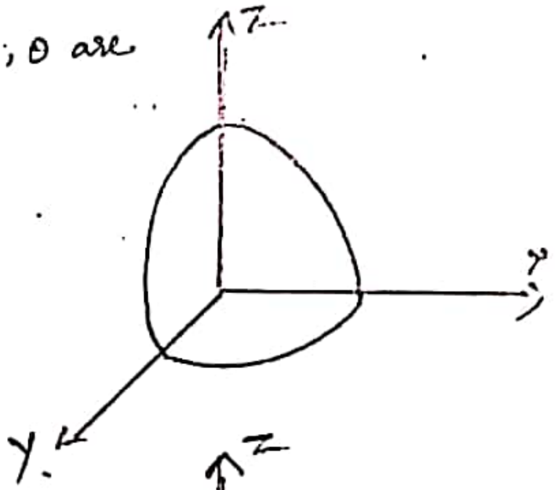
If the region of integration is a sphere $x^2+y^2+z^2=a^2$ with centre at $(0,0,0)$ and radius a, then limits of $r, \phi; \theta$ are

(a) Positive Octant of a sphere

r : $r=0$ to $r=a$.

θ : $\theta=0$ to $\theta=\pi/2$

ϕ : $\phi=0$ to $\phi=\frac{\pi}{2}$

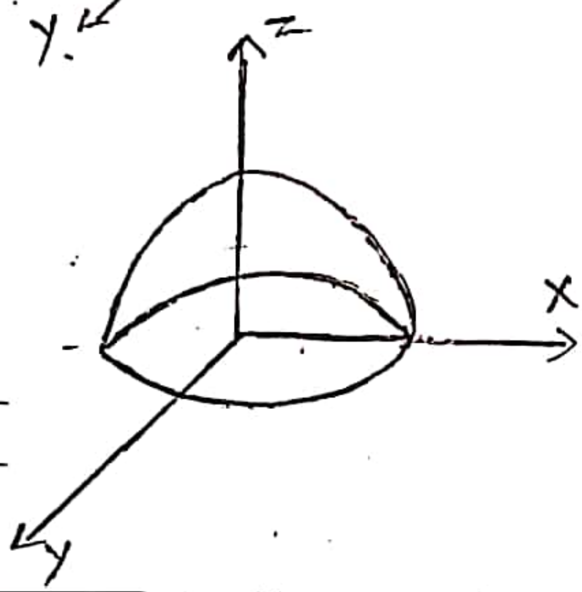


(b) Hemisphere (Above xy-plane i.e $z>0$)

r : $r=0$ to $r=a$.

θ : $\theta=0$ to $\theta=\pi/2$

ϕ : $\phi=0$ to $\phi=2\pi$



→ The range of angle θ is $0 \le \theta \le \pi$

→ The range of angle ϕ is $0 \le \phi \le 2\pi$

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→ Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar co ordinates.

Sol:- Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$, $f(x,y,z) = \frac{1}{\sqrt{1-x^2-y^2-z^2}}$

changing to spherical polar co ordinates by putting $x = r\sin\theta\cos\phi$.

$y = r\sin\theta\sin\phi$ $z = r\cos\theta$.

We have $J = r^2\sin\theta$, $x^2+y^2+z^2 = r^2$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

Also the given region of Integration is the volume of the sphere $x^2+y^2+z^2=1$ in the positive octant for which r varies from 0 to 1, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/2} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\,dr\,d\theta\,d\phi$$

$\left[\because z = \sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 1\right.$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/2} \left\{ \int_{r=0}^{r=1} \frac{1-(1-r^2)}{\sqrt{1-r^2}}\,dr \right\} \sin\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1} \left\{ \frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2} \right\} dr.\ \sin\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=\pi/2} \left[\sin^{-1}(r) - \left\{ \frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1}(r) \right\} \right]_{r=0}^{r=1} \sin\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=\pi/2} \left(\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{2} \right) \sin\theta\,d\theta\,d\phi.$$

$$= \frac{\pi}{4} \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=\pi/2} \sin\theta\,d\theta\,d\phi.$$

$$= \frac{\pi}{4} \int_{\phi=0}^{\phi=\pi/2} \left[-\cos\theta \right]_{\theta=0}^{\theta=\pi/2} d\phi$$

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$$= \frac{\pi}{4} \int_{\phi=0}^{\phi=\pi/2} \left[-\cos\frac{\pi}{2} + \cos 0\right] d\phi$$

$$= \frac{\pi}{4} \int_{\phi=0}^{\phi=\pi/2} d\phi$$

$$= \frac{\pi}{4} \left[\phi\right]_{\phi=0}^{\phi=\pi/2}$$

$$= \frac{\pi}{4} \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi^2}{8}.$$

→ Evaluate $\iiint (x^2+y^2+z^2)\, dx\, dy\, dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical polar coordinates.

Sol:- Let $I = \iiint (x^2+y^2+z^2)\, dx\, dy\, dz$, $f(x,y,z) = x^2+y^2+z^2$

Converting the given integral into spherical polar coordinates by putting

$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$.

We have $x^2+y^2+z^2 = r^2$, $J = r^2\sin\theta$.

Here r varies from 0 to a, θ varies from 0 to π and ϕ varies from 0 to 2π.

$$\therefore I = \iiint (x^2+y^2+z^2)\, dx\, dy\, dz = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} \cancel{r^2} \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi} d\phi \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta \int_{r=0}^{r=a} r^4\, dr$$

$$= \left[\phi\right]_{\phi=0}^{\phi=2\pi} \left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} \left[\frac{r^5}{5}\right]_{r=0}^{r=a}$$

$$= (2\pi - 0)(-\cos\pi + \cos 0)\left(\frac{a^5}{5} - 0\right)$$

$$\iiint (x^2+y^2+z^2)\, dx\, dy\, dz = \frac{4\pi a^5}{5}$$

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→ Using spherical polar coordinates evaluate $\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ taken over the volume bounded by the sphere $x^2+y^2+z^2=a^2$.

Sol: Let $I = \iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ $\quad f(x,y,z) = \frac{1}{x^2+y^2+z^2}$

changing to spherical polar coordinates by putting $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi \quad z = r\cos\theta$

We have $J = r^2\sin\theta \quad x^2+y^2+z^2 = r^2$ and $dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\phi$.

Also the given region of integration is the volume of the sphere $x^2+y^2+z^2=a^2$

Here r varies from 0 to a, θ varies from 0 to π and ϕ varies from 0 to 2π.

$$\therefore I = \iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2} = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \frac{r^2\sin\theta}{r^2}\, dr\,d\theta\,d\phi$$

$$= \int_{r=0}^{r=a} dr \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta \int_{\phi=0}^{\phi=2\pi} d\phi.$$

$$= [r]_{r=0}^{r=a}\,[-\cos\theta]_{\theta=0}^{\theta=\pi}\,[\phi]_{\phi=0}^{\phi=2\pi}$$

$$= (a-0)[-\cos\pi+\cos 0](2\pi-0)$$

$$\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2} = 4\pi a.$$

→ Using spherical polar coordinates, evaluate $\iiint xyz\,dx\,dy\,dz$ taken over the volume bounded by the sphere $x^2+y^2+z^2=a^2$ in the first octant.

Sol:- Let $I = \iiint xyz\,dx\,dy\,dz$ $\quad f(x,y,z) = xyz$

changing to spherical polar coordinates by putting $x = r\sin\theta\cos\phi$.

$y = r\sin\theta\sin\phi \quad z = r\cos\theta$

We have $J = r^2\sin\theta \quad x^2+y^2+z^2 = r^2$ and $dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\phi$.

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Also the given region of integration is the volume of the sphere $x^2+y^2+z^2=a^2$ in the positive octant.

Here r varies from 0 to a

θ varies from 0 to $\frac{\pi}{2}$

ϕ varies from 0 to $\frac{\pi}{2}$.

$$I = \iiint xyz\, dx\,dy\,dz = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi/2}\int_{\phi=0}^{\phi=\pi/2} r\sin\theta\cos\phi\; r\sin\theta\sin\phi\; r\cos\theta\; r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= \int_{r=0}^{r=a} r^5\,dr \int_{\theta=0}^{\theta=\pi/2} \sin^3\theta\cos\theta\,d\theta \int_{\phi=0}^{\phi=\pi/2} \sin\phi\cos\phi\,d\phi$$

$$= \left[\frac{r^6}{6}\right]_{r=0}^{r=a} \left[\frac{\sin^4\theta}{4}\right]_{\theta=0}^{\theta=\pi/2} \left[\frac{\sin^2\phi}{2}\right]_{\phi=0}^{\phi=\pi/2}$$

$$= \left[\frac{a^6}{6}-0\right]\left[\frac{\sin^4(\pi/2)}{4}-0\right]\left[\frac{\sin^2(\pi/2)}{2}-0\right]$$

$$\iiint xyz\,dx\,dy\,dz = \frac{a^6}{48}$$

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→ Using spherical polar coordinates find the volume of the sphere. $x^2+y^2+z^2=a^2$.

Sol:- Given that $x^2+y^2+z^2=a^2$.

Changing to spherical polar coordinates by putting $x=r\sin\theta\cos\phi$.

$y=r\sin\theta\sin\phi$ $z=r\cos\theta$.

We have $J=r^2\sin\theta$, $x^2+y^2+z^2=r^2$ and $dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\phi$.

Here θ varies from 0 to π.

ϕ varies from 0 to 2π

r varies from 0 to a.

$\therefore$ Volume $v=\iiint dx\,dy\,dz$

$$v=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\int_{r=0}^{r=a} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_{\theta=0}^{\theta=\pi}\sin\theta\,d\theta\cdot\int_{\phi=0}^{\phi=2\pi}d\phi\cdot\int_{r=0}^{r=a}r^2\,dr$$

$$=\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi}\left[\phi\right]_{\phi=0}^{\phi=2\pi}\left[\frac{r^3}{3}\right]_{r=0}^{r=a}$$

$$=\left[-\cos\pi+\cos 0\right]\left[2\pi-0\right]\left[\frac{a^3}{3}-0\right]$$

Volume $V=\dfrac{4\pi a^3}{3}$

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→ Using cylindrical co ordinates find the volume of the cylinder with base radius a and height h.

Sol: The region of Integration is bounded by $x^2+y^2 \le a^2$, $0 \le z \le h$.

The same region in cylindrical co ordinates will be as follows :

r varies from 0 to a.

θ varies from 0 to 2π.

z varies from 0 to h. and $dx\,dy\,dz = r\,dr\,d\theta\,dz$

$\therefore$ Volume $V = \iiint dx\,dy\,dz$

$$= \int_{z=0}^{z=h} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} r\,dr\,d\theta\,dz$$

$$= \int_{z=0}^{z=h} dz \int_{\theta=0}^{\theta=2\pi} d\theta \int_{r=0}^{r=a} r\,dr$$

$$= [z]_{z=0}^{z=h} \, [\theta]_{\theta=0}^{\theta=2\pi} \, \left[\frac{r^2}{2}\right]_{r=0}^{r=a}$$

$$(h-0)\,(2\pi-0)\left(\frac{a^2}{2}-0\right)$$

Volume $V = \pi a^2 h$

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→ Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$

Sol:- Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$

In the given integral the inner integral limits are interms of x and y. so these are limits of z.

$\therefore$ z limits $z = \sqrt{x^2+y^2}$, $z = 1$. i.e equation of cone $z^2 = x^2+y^2$.

The second integral limits are interms of x so these are limits of y.

$\therefore$ y limits $y = 0$, $y = \sqrt{1-x^2} \Rightarrow x^2+y^2 = 1$.

The outer Integral limits are constants, these are limits of x.

x limits $x = 0$, $x = 1$.

Clearly the region of integrations is bounded by the cone $z^2 = x^2+y^2$ and the cylinder $x^2+y^2 = 1$, the plane $z = 1$ in the positive octant of the plane

It is difficult to integrate in cartesian form, so we can solve by spherical coordinate system.

Now, using spherical coordinate system.

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $x^2+y^2+z^2 = r^2$

$\tan\phi = \frac{y}{x}$ $\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$ $dx\,dy\,dz = r^2 \sin\theta \, dr\,d\theta\,d\phi$.

Limits in spherical coordinate form.

(i) $z = r\cos\theta$

$1 = r\cos\theta \Rightarrow r = \sec\theta$.

$\therefore 0 \leq r \leq \sec\theta$.

(ii) $\tan\theta = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{z^2}}{z} = 1 \Rightarrow \theta = \frac{\pi}{4}$.

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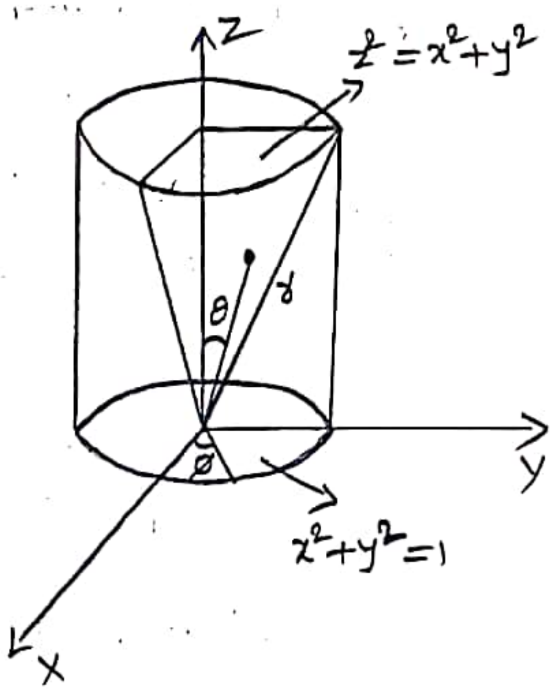
iii) At $y=0 \Rightarrow \phi = \tan^{-1}\left(\frac{0}{x}\right) = 0$.

and at $x=0 \Rightarrow \phi = \tan^{-1}\left(\frac{y}{0}\right) = \frac{\pi}{2}$

$\therefore$ Limits in spherical coordinate system are



Limits of r : $r=0$ to $r=\sec\theta$

Limits of θ : $\theta=0$ to $\theta=\frac{\pi}{4}$

Limits of ϕ : $\phi=0$ to $\phi=\frac{\pi}{2}$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}} = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sec\theta} \frac{1}{\sqrt{r^2}} r^2 \sin\theta \, dr\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=\sec\theta} r\sin\theta \, dr\,d\theta\,d\phi.$$

$$= \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=\pi/4} \sin\theta \left[\frac{r^2}{2}\right]_{r=0}^{r=\sec\theta} d\theta\,d\phi.$$

$$= \frac{1}{2} \int_{\phi=0}^{\phi=\pi/2} \int_{\theta=0}^{\theta=\pi/4} \sin\theta \sec^2\theta \, d\theta\,d\phi$$

$$= \frac{1}{2} \int_{\phi=0}^{\phi=\pi/2} d\phi \int_{\theta=0}^{\theta=\pi/4} \sec\theta \tan\theta \, d\theta$$

$$= \frac{1}{2} \left[\phi\right]_{\phi=0}^{\phi=\pi/2} \left[\sec\theta\right]_{\theta=0}^{\theta=\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-0\right)\left[\sec\frac{\pi}{4}-1\right]$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}} = \frac{\pi}{4}\left(\sqrt{2}-1\right).$$

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If R is the region in the 1^{st} octant bounded by the sphere $x^2+y^2+z^2=a^2$ Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$ by changing it to spherical polar co ordinates ②

Sol: Let $I = \iiint_R (x+y+z)\,dx\,dy\,dz$

To transform it into spherical polar co ordinates

put $x = \sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi \qquad J = r^2\sin\theta$

In the given region, r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \iiint_R (x+y+z)\,dx\,dy\,dz = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\phi=0}^{\phi=\pi/2} (r\sin\theta\cos\phi + r\sin\theta\sin\phi + r\cos\theta)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{r=0}^{r=a} r^3\,dr \int_{\theta=0}^{\theta=\pi/4}\int_{\phi=0}^{\phi=\frac{\pi}{2}} \left[\sin^2\theta(\cos\phi+\sin\phi) + \sin\theta\cos\theta\right] d\phi\,d\theta$$

$$= \left[\frac{r^4}{4}\right]_0^a \int_{\theta=0}^{\theta=\pi/2} \left[\sin^2\theta\,(\sin\phi-\cos\phi) + \sin\theta\cos\theta\,[\phi]\right]_{\phi=0}^{\phi=\pi/2} d\theta$$

$$= \frac{a^4}{4}\int_{\theta=0}^{\theta=\pi/2}\left[2\sin^2\theta + \frac{\pi}{2}\sin\theta\cos\theta\right]d\theta$$

$$= \frac{a^4}{4}\,2.\int_{\theta=0}^{\theta=\pi/2}\left[(1-\cos 2\theta) + \frac{\pi}{2}\sin\theta\cos\theta\right]d\theta$$

$$= \frac{a^4}{4}\left[\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{\pi}{2}\,\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^4}{4}\left[\frac{\pi}{2}+\frac{\pi}{4}\right] = \frac{a^4}{4}\left(\frac{2\pi+\pi}{4}\right)$$

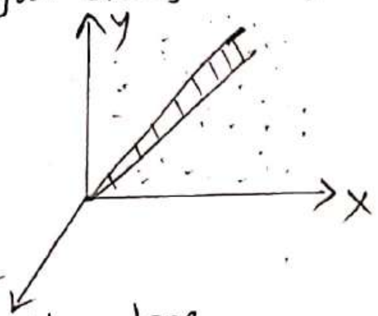
$$= \frac{3\pi a^4}{16}.$$

Evaluate $\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$ (3)

Sol:- Let $I = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$, $f(x,y,z) = \frac{1}{(1+x^2+y^2+z^2)^2}$

It is difficult to integrate this integral in cartesian form.

Put $x = r\cos\phi\sin\theta$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ integral changes to spherical form. $|J| = r^2\sin\theta$, $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$.



→ Limits of x : $x = 0$ to $x \to \infty$

Limits of y : $y = 0$ to $y \to \infty$

Limits of z : $z = 0$ to $z \to \infty$

The region of integration is the positive octant of the plane

Limits of r : $r = 0$ to $r \to \infty$.

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Limits of ϕ : $\phi = 0$ to $\phi = \frac{\pi}{2}$

Hence, the spherical form of the given integral is

$$I = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2} = \int_{r=0}^{r\to\infty}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(1+r^2)^2}$$

$$= \int_{r=0}^{r\to\infty} \frac{r^2}{(1+r^2)^2}\,dr \int_{\theta=0}^{\theta=\pi/2} \sin\theta\,d\theta \int_{\phi=0}^{\phi=\pi/2} d\phi.$$

$$I = \int_{\theta=0}^{\theta=\pi/2} \sin\theta\,d\theta \int_{t=0}^{t=\pi/2} \frac{\tan^2 t}{\sec^4 t}\sec^2 t\,dt \int_{\phi=0}^{\phi=\pi/2} d\phi$$

Put $r = \tan t$
$dr = \sec^2 t\,dt$
when $r = 0, t = 0$
When $r \to \infty$, $t = \frac{\pi}{2}$

$$= \left[-\cos\theta\right]_0^{\pi/2} \int_0^{\pi/2} \sin^2 t\,dt \,\left[\phi\right]_0^{\pi/2}$$

$$= \left[-\cos\frac{\pi}{2} + \cos 0\right] \int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)dt \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\left(t - \frac{\sin 2t}{2}\right)_0^{\pi/2}$$

$$= \frac{\pi}{4}\left[\left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right) - 0\right]$$

$$= \frac{\pi^2}{8}$$

Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{1/2}}$ over the region bounded by the spheres (4)

$x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $a>b>0$

Sol. Let $I = \iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{1/2}}$, $f(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{1/2}}$

Converting the given integral into spherical polar co ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$; $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$.

$J = r^2\sin\theta$.

Equations of the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ reduce to $r=a$ and $r=b$ respectively.

Draw an elementary radius vector OAB from the origin in the region

This radius vector enter in the region from the sphere $r=b$ and terminates on the sphere $r=a$.

Limits of r: $r=b$ to $r=a$.

For the complete sphere Limits of θ, $\theta=0$ to $\theta=\pi$

Limits of ϕ, $\phi=0$ to $\phi=2\pi$

Hence the spherical form of the given integral is.

$$I = \iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{1/2}}$$

$$= \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=b}^{r=a} \frac{r^2\sin\theta}{r}\,dr\,d\theta\,d\phi.$$

$$= \int_{\phi=0}^{\phi=2\pi} d\phi \int_{\theta=0}^{\theta=\pi}\sin\theta\,d\theta \int_{r=b}^{r=a} r\,dr$$

$$= [\phi]_0^{2\pi}\,[-\cos\theta]_0^{\pi}\,\left[\frac{r^2}{2}\right]_b^a$$

$$= (2\pi-0)(-\cos\pi+\cos 0)\left(\frac{a^2-b^2}{2}\right)$$

$$= 2\pi(a^2-b^2)$$

VECTOR CALCULUS.

Vector Differentiation.

Differentiation of a Vector Function :-

Let S be a set of real numbers. Corresponding to each scalar $t \in S$, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a

vector (vector valued) function. S is called the domain of $\bar{f}$.

We write $\bar{f} = \bar{f}(t)$.

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimen-

sional space. We can write $\bar{f} = \bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$

Where $f_1(t)\ f_2(t)\ f_3(t)$ are real valued functions (which are called components of $\bar{f}$)

Derivative :-

Let $\bar{f}$ be a vector function on an interval I and $a \in I$. Then

$\underset{t \to a}{Lt} \dfrac{\bar{f}(t) - \bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is

denoted by $\bar{f}'(a)$ (or) $\left(\dfrac{d\bar{f}}{dt}\right)$ at $t=a$. We also say that $\bar{f}$ is differen-

tiable at $t=a$. If $\bar{f}'(a)$ exists.

Higher Order Derivatives :-

Let $\bar{f}$ be differentiable on an interval I and $\bar{f}' = \dfrac{d\bar{f}}{dt}$ be the derivative

of $\bar{f}$. If $\underset{t \to a}{Lt} \dfrac{\bar{f}'(t) - \bar{f}'(a)}{t-a}$ exists for every $a \in I_1 \subset I$ then $\bar{f}'$ is said

to be differentiable on I_1. It is denoted by $\bar{f}''(a)$ (or) $\dfrac{d^2\bar{f}}{dt^2}$.

Similarly we can define $\bar{f}'''(t)$ etc.

Properties :-

(i) Derivative of a constant vector is $\bar{0}$

(ii) The necessary and sufficient condition for $\bar{f}(t)$ to be constant vector function is $\dfrac{d\bar{f}}{dt} = \bar{0}$

(iii) If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then .

(a) $\frac{d}{dt}(\bar{a} \pm \bar{b}) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$

(b) $\frac{d}{dt}(\bar{a} \cdot \bar{b}) = \frac{d\bar{a}}{dt} \cdot \bar{b} + \bar{a} \cdot \frac{d\bar{b}}{dt}$.

(c) $\frac{d}{dt}(\bar{a} \times \bar{b}) = \frac{d\bar{a}}{dt} \times \bar{b} + \bar{a} \times \frac{d\bar{b}}{dt}$.

(iv) If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differen-tial function, then $\frac{d}{dt}(\phi \bar{f}) = \phi \frac{d\bar{f}}{dt} + \frac{d\phi}{dt}\bar{f}$.

(v) If $\bar{f} = f_1(t)\, i + f_2(t)\, j + f_3(t)\, k$ then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt} i + \frac{df_2}{dt} j + \frac{df_3}{dt} k$.

Where $f_1(t)$, $f_2(t)$, $f_3(t)$ are cartesian components of the vector $\bar{f}$.

Partial Derivatives :—

Let $\bar{f}$ be a vector function of scalar variables p, q, t. Then we write $\bar{f} = \bar{f}(p, q, t)$. Treating t as a variable and p, q as constants, we define. $\underset{\delta t \to 0}{Lt} \frac{\bar{f}(p, q, t+\delta t) - \bar{f}(p, q, t)}{\delta t}$ if exists, as partial derivative of $\bar{f}$ w.r.t."t" and is denoted by $\frac{\partial \bar{f}}{\partial t}$.

Similarly we can define $\frac{\partial \bar{f}}{\partial p}$, $\frac{\partial \bar{f}}{\partial q}$ also ..

Properties :—

(i) $\frac{\partial}{\partial t}(\phi \bar{a}) = \frac{\partial \phi}{\partial t}\bar{a} + \phi \frac{\partial \bar{a}}{\partial t}$.

(ii) If λ is a constant, then $\frac{\partial}{\partial t}(\lambda \bar{a}) = \lambda \frac{\partial \bar{a}}{\partial t}$.

(iii) If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi \bar{c}) = \bar{c}\frac{\partial \phi}{\partial t}$.

(iv) $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$.

(v) $\frac{\partial}{\partial t}(\bar{a} \cdot \bar{b}) = \frac{\partial \bar{a}}{\partial t} \cdot \bar{b} + \bar{a} \cdot \frac{\partial \bar{b}}{\partial t}$.

(vi) $\frac{\partial}{\partial t}(\bar{a} \times \bar{b}) = \frac{\partial \bar{a}}{\partial t} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial t}$

(vii) Let $\bar{f} = f_1 i + f_2 j + f_3 k$ where f_1, f_2, f_3 are differentiable scalar functions of more than one variable. Then $\frac{\partial \bar{f}}{\partial t} = i\frac{\partial f_1}{\partial t} + j\frac{\partial f_2}{\partial t} + k\frac{\partial f_3}{\partial t}$.

Vector Point function and vector field :-

Let P be any point in a region 'D' of space. Let γ be the position vector of P. If there exists a vector function F corresponding to each P, then such a function F is called a vector point function and the region D is called a vector field.

Eg:- Consider the vector function. $\bar{F} = (x-y)\,i + xy\,j + yz\,k$ —— ①.

Let P be a point whose position vector is $\gamma = 2i + j + 3k$ in the region D of space.

At P, the value of F is obtained by putting $x=2,\ y=1,\ z=3$ in $\bar{F}$

i.e At P, $F = i + 2j + 3k$.

Thus, to each point P of the region D, there corresponds a vector F given by the vector function ①.

Hence F is a vector point function (of scalar variables, x, y, z) and the region D is a vector field.

Eg:- Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is a vector point function. Similarly, the acceleration of the particle is also a vector point function.

Eg:- In a magnetic field at any point $P(x,y,z)$, there will be a magnetic force $\bar{F}(x,y,z)$. This is called magnetic force field.

Scalar point function and scalar field :—

If there exists a scalar f given by a scalar function f correspon-ding to each point P (with position vector r) in a region D of space. f is called a scalar point function and D is called a scalar field.

Eg:- Let P be a point whose position vector is $r = 2i + j + 3k$.

Consider $f = xyz + xy + z$.

Then the value of f at P is obtained by putting $x=2$, $y=1$, $z=3$.

i.e At P, $f = 2.1.3 + 2.1 + 3 = 11$.

Hence the scalar 11 is attached to the point P.

The function f is a scalar point function (of scalar variables, x, y, z) and D is a scalar field.

Eg:- Consider a heated solid. At each point $P(x,y,z)$ of the solid, there will be temparature $T(x,y,z)$. This T is a scalar point function.

Eg:- Suppose a particle (or a very small insect) is tracing a path in space. when it occupies a position $P(x,y,z)$ in space, it will be having some speed say, v. This speed v is a scalar point function.

Vector Differential Operator :-

The vector differential operator ∇ (read as del) is defined as

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$. ($i, j, k$ are unit vectors in x, y, z directions)

→ This operator ∇ is used in defining the gradient, divergence and curl.

→ Properties of ∇ are similar to those of vectors. The operator is applied to both vectors and scalar functions.

Gradient of a scalar point function :-

Let $\phi(x, y, z)$ be a scalar point function of position defined in some region of space. Then the vector function $i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by grad ϕ or $\nabla\phi$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

Properties :-

(i) If f and g are two scalar functions then $\text{grad}(f \pm g) = \text{grad} f \pm \text{grad} g$.

(ii) $\text{grad}(fg) = f(\text{grad} g) + g(\text{grad} f)$.

(iii) If c is a constant, $\text{grad}(cf) = c(\text{grad} f)$

(iv) $\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad} f) - f(\text{grad} g)}{g^2}, (g \neq 0)$.

(v) Let $\bar{r} = xi + yj + zk$ Then $d\bar{r} = dx\, i + dy\, j + dz\, k$. If ϕ is any scalar point function, then $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$d\phi = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right).(i\,dx + j\,dy + k\,dz)$$

$$d\phi = \nabla\phi . d\bar{r}$$

(vi) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla\phi = \bar{0}$

(vii) If ϕ defines a scalar field 'gradϕ' or '$\nabla\phi$' defines a vector field.

Physical significance of gradϕ :—

If $\phi(x,y,z) = c$ (c being a constant) represents a surface, then 'gradϕ' represents the normal vector to the surface at the point (x,y,z).

For, if $r = xi + yj + zk$ is the position vector of the point (x,y,z) on the surface, we have $dr = dx\, i + dy\, j + dz\, k$ which is in the tangent plane to the surface of $\phi(x,y,z)$.

$$\text{Again } \nabla\phi \cdot dr = \left[\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right] \cdot (dx\, i + dy\, j + dz\, k)$$

$$= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz .$$

$$= d\phi \quad [\because \phi = c].$$

$\therefore$ The vector $\nabla\phi$ which is perpendicular to the tangent plane is the normal vector to $\phi = c$ at (x,y,z).

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Note :- If $\phi(x,y,z) = c$ represents a surface.

(i) Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

(ii) Unit normal vector to the surface ϕ is given by.

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

1) Find the unit normal vector to the surface $z = x^2+y^2$ at the point $(1,-2,5)$

Sol:- Let the surface be $\phi = x^2+y^2-z$.

Given that the point $P(1,-2,5)$.

Normal vector to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial x} = 2x \quad \frac{\partial\phi}{\partial y} = 2y \quad \frac{\partial\phi}{\partial z} = -1.$$

At the point $P(1,-2,5)$

$$\frac{\partial\phi}{\partial x} = 2 \quad \frac{\partial\phi}{\partial y} = -4 \quad \frac{\partial\phi}{\partial z} = -1$$

$$\therefore \nabla\phi = 2i - 4j - k$$

Unit normal vector to the surface ϕ is given by.

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i-4j-k}{\sqrt{2^2+(-4)^2+(-1)^2}} = \frac{2i-4j-k}{\sqrt{21}}.$$

2) Find a unit normal vector to the surface $x^2yz+4xz^2$ at $(1,-2,-1)$.

Sol:- Let the surface be $\phi = x^2yz+4xz^2$.

Given that the point $P(1,-2,-1)$

Normal vector to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = 2xyz+4z^2 \quad \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y+8xz$$

At the point $P(1,-2,-1)$ $\quad \frac{\partial\phi}{\partial x} = 8, \frac{\partial\phi}{\partial y} = -1 \quad \frac{\partial\phi}{\partial z} = -10$

$$\therefore \nabla\phi = 8i - j - 10k.$$

Unit normal vector to the surface ϕ is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8i - j - 10k}{\sqrt{64+1+100}} = \frac{8i - j - 10k}{\sqrt{165}}$$

If $f = x+y+z$, $g = x^2+y^2+z^2$, $h = xy+yz+zx$. prove that $[\text{grad}f\ \text{grad}g\ \text{grad}h] = 0$

sol: Given that $f = x+y+z$ $\quad g = x^2+y^2+z^2$ $\quad h = xy+yz+zx$.

$$\text{grad}f = \frac{\partial f}{\partial x} i + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial y} = 1 \quad \frac{\partial f}{\partial z} = 1$$

$$\therefore \text{grad}f = i + j + k.$$

$$\text{grad}g = i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 2x \quad \frac{\partial g}{\partial y} = 2y \quad \frac{\partial g}{\partial z} = 2z$$

$$\text{grad}g = 2x i + 2y j + 2z k.$$

$$\text{grad}h = i\frac{\partial h}{\partial x} + j\frac{\partial h}{\partial y} + k\frac{\partial h}{\partial z}.$$

$$\frac{\partial h}{\partial x} = y+z \quad \frac{\partial h}{\partial y} = x+z \quad \frac{\partial h}{\partial z} = x+y$$

$$\text{grad}h = (y+z)i + (x+z)j + (x+y)k.$$

$$[\text{grad}f\ \text{grad}g\ \text{grad}h] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2x & 2(y-x) & 2(z-x) \\ y+z & -y+x & x-z \end{vmatrix}$$

$$= y-x)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 2 & 2 \\ y+z & -1 & -1 \end{vmatrix} = 0$$

$$[\text{grad}f\ \text{grad}g\ \text{grad}h] = 0.$$

Angle between two surfaces :-

Let $\phi_1(x,y,z)$ and $\phi_2(x,y,z)$ be two surfaces.

We know that the angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Normal vector to the surface ϕ_1 is given by.

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}.$$

Normal vector to the surface ϕ_2 is given by

$$\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}.$$

Let $\bar{a} = \nabla\phi_1 \qquad \bar{b} = \nabla\phi_2$.

Let θ be the angle between the two surfaces Then.

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}.$$

Note :-

(i) If $\theta = \frac{\pi}{2}$, $\quad \bar{a}\cdot\bar{b} = 0$ Then two surfaces ϕ_1 and ϕ_2 are $\perp$er.

(ii) If $\theta = 0$, $\quad \bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|$.

Find the acute angle between the surface $xy^2z = 2$ and $x^2+y^2+z^2=6$ at the point $(2,1,1)$.

Sol:- Let $\phi_1 = xy^2z - 2$ and $\phi_2 = x^2+y^2+z^2-6$.

We know that the angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Normal to the surface ϕ_1 is $\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$.

$$\frac{\partial\phi_1}{\partial x} = y^2z \qquad \frac{\partial\phi_1}{\partial y} = 2xyz \qquad \frac{\partial\phi_1}{\partial z} = xy^2$$

At the point $(2,1,1)$ $\quad \frac{\partial\phi_1}{\partial x} = 1 \quad \frac{\partial\phi_1}{\partial y} = 4 \quad \frac{\partial\phi_1}{\partial z} = 2$

$$\bar{a} = \nabla\phi_1 = i+4j+2k.$$

Normal to the surface ϕ_2 is $\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$.

$$\frac{\partial\phi_2}{\partial x} = 2x \qquad \frac{\partial\phi_2}{\partial y} = 2y \qquad \frac{\partial\phi_2}{\partial z} = 2z.$$

At the point $(2,1,1)$ $\quad \frac{\partial\phi_2}{\partial x} = 4 \quad \frac{\partial\phi_2}{\partial y} = 2 \quad \frac{\partial\phi_2}{\partial z} = 2$

$$\bar{b} = \nabla\phi_2 = 4i+2j+2k.$$

The vectors $\bar{a}$ and $\bar{b}$ are along the normals to the surfaces ϕ_1 and ϕ_2 at the pt $(2,1,1)$.

Let θ be the angle between two surfaces. Then

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$$

$$\cos\theta = \left|\frac{(i+4j+2k)\cdot(4i+2j+2k)}{\sqrt{1+4^2+2^2}\sqrt{4^2+2^2+2^2}}\right| = \frac{4+8+4}{\sqrt{1+16+4}\sqrt{16+4+4}}$$

$$\cos\theta = \frac{16}{\sqrt{21}\sqrt{24}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{14}}\right).$$

Find the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$.

Sol:- Given that the surface $\phi = xy - z^2$.

WKT the normal to the surface ϕ is given by $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = y \quad \frac{\partial\phi}{\partial y} = x \quad \frac{\partial\phi}{\partial z} = -2z.$$

$$\nabla\phi = yi + xj - 2zk.$$

Let $\bar{a}$ and $\bar{b}$ be the normals to the surface ϕ at $(4,1,2)$ and $(3,3,-3)$

At the pt $(4,1,2)$

$$\bar{a} = i + 4j - 4k.$$

At the pt $(3,3,-3)$

$$\bar{b} = 3i + 3j + 6k.$$

Let θ be the angle between the two normals.

$$\therefore \quad \cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$\cos\theta = \frac{(i+4j-4k)\cdot(3i+3j+6k)}{\sqrt{1+16+16}\,\sqrt{9+9+36}}$$

$$\cos\theta = \frac{3+12-24}{\sqrt{33}\,\sqrt{54}} = \frac{-9}{\sqrt{33}\,\sqrt{54}}$$

Find the constants p and q such that the surfaces $px^2 - qyz = (p+2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point $(1,-1,2)$.

sol: Let $\phi_1 = px^2 - qyz - (p+2)x$ —— ①

$\phi_2 = 4x^2y + z^3 - 4$. —— ②

Given that the surfaces ϕ_1 and ϕ_2 are orthogonal at the point $(1,-1,2)$

Since the point $(1,-1,2)$ lies on ① and ②.

We have $p + 2q - p - 2 = 0$ [$\because$ from ①]

$q = 1$.

Normal to the surface ϕ_1 is given by $\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$

$\frac{\partial\phi_1}{\partial x} = 2px - p - 2$ $\quad \frac{\partial\phi_1}{\partial y} = -qz$ $\quad \frac{\partial\phi_1}{\partial z} = -qy$.

$\nabla\phi_1 = (2px - p - 2)i - qzj - qyk$.

At the point $(1,-1,2)$

$\nabla\phi_1 = (p-2)i - 2qj + qk = \bar{a}$ (say)

Normal to the surface ϕ_2 is given by $\nabla\phi_2 = i\frac{\partial\phi_2}{\partial x} + j\frac{\partial\phi_2}{\partial y} + k\frac{\partial\phi_2}{\partial z}$.

$\frac{\partial\phi_2}{\partial x} = 8xy$ $\quad \frac{\partial\phi_2}{\partial y} = 4x^2$ $\quad \frac{\partial\phi_2}{\partial z} = 3z^2$:

$\nabla\phi_2 = (8xy)i + (4x^2)j + (3z^2)k$

At the point $(1,-1,2)$

$\nabla\phi_2 = -8i + 4j + 12k = \bar{b}$ (say)

Angle between the surfaces is given by $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$.

Since the surfaces ① and ② are orthogonal (Perpendicular) i.e $\theta = \frac{\pi}{2}$

Then $\bar{a}\cdot\bar{b} = 0$

$[(p-2)i - 2qj + qk]\cdot[-8i + 4j + 12k] = 0$.

$$-8(p-2)+4(-2q)+12q=0$$

$$-8p+16-8q+12q=0$$

$$2p-q=4$$

$$2p-1=4 \quad [\because q=1]$$

$$p=\frac{5}{2}$$

$\therefore$ The values of p and q are $p=\frac{5}{2}$, $q=1$.

Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$.

sol:- Let $f=x^2+y^2+z^2-29$ $\quad g=x^2+y^2+z^2+4x-6y-8z-47$.

WKT the angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Normal to the surface f is given by $\nabla f = i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x}=2x \quad \frac{\partial f}{\partial y}=2y \quad \frac{\partial f}{\partial z}=2z$$

At the pt $(4,3,2)$ = $\frac{\partial f}{\partial x}=8 \quad \frac{\partial f}{\partial y}=-6 \quad \frac{\partial f}{\partial z}=4$

$$\bar{a}=\nabla f = 8i-6j+4k.$$

Normal to the surface g is given by $\nabla g = i\frac{\partial g}{\partial x}+j\frac{\partial g}{\partial y}+k\frac{\partial g}{\partial z}$.

$$\frac{\partial g}{\partial x}=2x+4 \quad \frac{\partial g}{\partial y}=2y-6 \quad \frac{\partial g}{\partial z}=2z-8.$$

At the pt $(4,-3,2)$ $\quad \frac{\partial g}{\partial x}=12 \quad \frac{\partial g}{\partial y}=-12 \quad \frac{\partial g}{\partial z}=-4$

$$\bar{b}=\nabla g = 12i-12j-4k.$$

Let θ be the angle b/n the normals $\bar{a}$ and $\bar{b}$. Then

$$\cos\theta = \frac{\bar{a}.\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$\cos\theta = \frac{(8i-6j+4k).(12i-12j-4k)}{\sqrt{64+36+16}\,\sqrt{144+144+16}}$$

$$\cos\theta = \frac{15\cancel{4}}{\sqrt{116}\,\sqrt{304}}$$

$$\theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

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Find the angle between the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$.

Sol:- Let $\phi = x^2 - yz$.

Let $\bar{a}$ and $\bar{b}$ be the normals to this surface at the points $(1,1,1)$ and $(2,4,1)$ respectively.

Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial x} = 2x \qquad \frac{\partial\phi}{\partial y} = -z \qquad \frac{\partial\phi}{\partial z} = -y.$$

$$\therefore \quad \nabla\phi = (2x)i + (-z)j + (-y)k.$$

At the point $P_1(1,1,1)$ $\quad \bar{a} = \nabla\phi = 2i - j - k.$

At the point $P_2(2,4,1)$ $\quad \bar{b} = \nabla\phi = 4i - j - 4k$

Let θ be the angle between the normals $\bar{a}$ and $\bar{b}$. Then

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$$

$$\cos\theta = \frac{(2i - j - k)\cdot(4i - j - 4k)}{\sqrt{4+1+1}\,\sqrt{16+16+1}}$$

$$\cos\theta = \frac{8+1+4}{\sqrt{6}\,\sqrt{33}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right)$$

Directional Derivative :-

Let $\phi(x,y,z)$ be a scalar function defined throughout some region of space.

If $\bar{a}$ be any vector. $\frac{\nabla\phi.\bar{a}}{|\bar{a}|}$ which represents the component of $\nabla\phi$ in the direction of $\bar{a}$ is known as the directional derivative of ϕ in the direction of $\bar{a}$.

Note :- (i) The Directional Derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi.\frac{\bar{a}}{|\bar{a}|}$

(ii) Physically the directional derivative is the rate of change of ϕ in the direction of $\bar{a}$.

(iii) The directional derivative will be maximum in the direction of $\nabla\phi$ (i.e $\bar{a}=\nabla\phi$) and the maximum value of the directional derivative

$$\nabla\phi.\frac{\bar{a}}{|\bar{a}|} = \nabla\phi.\frac{\nabla\phi}{|\nabla\phi|} = \frac{|\nabla\phi|^2}{|\nabla\phi|} = |\nabla\phi|$$

TYPE – 1

1) Find the directional derivative of $2xy+z^2$ at $(1,-1,3)$ in the direction of $i+2j+3k$. Also find maximum directional derivative.

Sol:- The directional derivative of ϕ in the direction of $\bar{a}$ is given by

$$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

Let $\phi = 2xy+z^2$, $\quad \bar{a} = i+2j+3k$.

WKT $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial x} = 2y \qquad \frac{\partial\phi}{\partial y} = 2x \qquad \frac{\partial\phi}{\partial z} = 2z$$

At the point $(1,-1,3)$, $\frac{\partial\phi}{\partial x} = -2 \quad \frac{\partial\phi}{\partial y} = 2 \quad \frac{\partial\phi}{\partial z} = 6$.

$\therefore \quad \nabla\phi = -2i+2j+6k$.

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (-2i+2j+6k) \cdot \frac{(i+2j+3k)}{\sqrt{1+4+9}}$$

$$= \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}.$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{(-2)^2+2^2+6^2}$

$$= \sqrt{4+4+36}$$

$$= \sqrt{44}.$$

2) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2i-j-2k$. Also find maximum directional derivative.

Ans:- $\frac{37}{3}$

Sol:- Given that $\phi = x^2yz + 4xz^2$.

Let $\bar{a} = 2i-j-2k$

The directional derivative of ϕ in the direction of $\bar{a}$ is given by

$$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

Wkt $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2$ $\quad \frac{\partial\phi}{\partial y} = x^2z$ $\quad \frac{\partial\phi}{\partial z} = x^2y + 8xz$

At the point $(1,-2,-1)$ $\quad \frac{\partial\phi}{\partial x} = 8$ $\quad \frac{\partial\phi}{\partial y} = -1$ $\quad \frac{\partial\phi}{\partial z} = -10$

$\therefore \nabla\phi = 8i - j - 10k.$

Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (8i - j - 10k) \cdot \frac{(2i - j - 2k)}{\sqrt{4+1+4}}$$

$$= \frac{16+1+20}{3}$$

$$= \frac{37}{3}.$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{8^2 + (-1)^2 + (-10)^2} = \sqrt{165}$

TYPE-2 :

1) Find the directional derivative of the function $x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where $Q(5,0,4)$

Sol:- WKT the directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = x^2-y^2+2z^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x \qquad \frac{\partial\phi}{\partial y} = -2y \qquad \frac{\partial\phi}{\partial z} = 4z$$

At the point $P(1,2,3)$ $\quad \frac{\partial\phi}{\partial x} = 2 \quad \frac{\partial\phi}{\partial y} = -4 \quad \frac{\partial\phi}{\partial z} = 12$

$$\therefore \nabla\phi = 2i-4j+12k.$$

The position vectors of P and Q with respect to the origin are

$\overline{OP} = i+2j+3k$ and $\overline{OQ} = 5i+0j+4k$.

$$\therefore \overline{PQ} = \overline{OQ}-\overline{OP} = 4i-2j+k.$$

Let $\bar{a} = 4i-2j+k$.

$$\therefore \text{Directional Derivative} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (2i-4j+12k) \cdot \frac{4i-2j+k}{\sqrt{16+4+1}}$$

$$= \frac{8+8+12}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}}$$

$\therefore$ Maximum Directional Derivative $= |\nabla\phi| = \sqrt{2^2+(-4)^2+12^2}$

$$= \sqrt{4+16+144}$$

$$= \sqrt{164}$$

2) Find the directional derivative of the scalar point function $4xy^2 + 2x^2yz$ at the point $A(1,2,3)$ in the direction of the line AB where $B(5,0,4)$. Also find maximum directional derivative. $\left[\text{Ans: } \frac{120}{\sqrt{21}}\right]$

Sol:- We know that the directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = 4xy^2 + 2x^2yz$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 4y^2 + 4xyz \quad \frac{\partial\phi}{\partial y} = 8xy + 2x^2z, \quad \frac{\partial\phi}{\partial z} = 2x^2y$$

At the point $P(1,2,3)$ $\quad \frac{\partial\phi}{\partial x} = 40 \quad \frac{\partial\phi}{\partial y} = 22 \quad \frac{\partial\phi}{\partial z} = 4$

$$\nabla\phi = 40i + 22j + 4k.$$

The position vectors of P and Q with respect to the origin are

$\overline{OP} = i + 2j + 3k, \quad \overline{OQ} = 5i + 0j + 4k.$

$$PQ = \overline{OQ} - \overline{OP} = 4i - 2j + k$$

$$\text{Let } \bar{a} = 4i - 2j + k$$

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (40i + 22j + 4k) \cdot \frac{(4i - 2j + k)}{\sqrt{16+4+1}}$$

$$= \frac{160 - 44 + 4}{\sqrt{21}}$$

$$= \frac{200}{\sqrt{21}}$$

$\therefore$ Maximum Directional Derivative $= |\nabla\phi| = \sqrt{(40)^2 + (22)^2 + 16}$

$$= \sqrt{1600 + 484 + 16}$$

$$= \sqrt{2100}.$$

Note :- Let the equation of the curve be $x = x_1(t)$ $y = y_1(t)$, $z = z_1(t)$ — ①

Let $\bar{r}$ be the position vector of any point on the curve ①.

Then $\bar{r} = x_1(t)\,i + y_1(t)\,j + z_1(t)\,k$. $\left[\because \bar{r} = xi + yj + zk\right]$.

$\frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve ①.

TYPE – 3

(1) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point $(1,1,1)$.

Sol:- Wkt. the directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

Let $\phi = xy^2 + yz^2 + zx^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$\frac{\partial\phi}{\partial x} = (y^2 + 2xz)$ $\quad \frac{\partial\phi}{\partial y} = 2xy + z^2$ $\quad \frac{\partial\phi}{\partial z} = 2yz + x^2$

At the pt $(1,1,1)$ $\quad \frac{\partial\phi}{\partial x} = 3$ $\quad \frac{\partial\phi}{\partial y} = 3$ $\quad \frac{\partial\phi}{\partial z} = 3$.

$$\therefore \nabla\phi = 3i + 3j + 3k.$$

Given that the curve $x = t$ $y = t^2$ $z = t^3$ — ①

Let $\bar{r}$ be the position vector of any point on the curve ①

$$\bar{r} = xi + yj + zk.$$

$$\bar{r} = ti + t^2 j + t^3 k.$$

$$\frac{d\bar{r}}{dt} = i + 2tj + 3t^2 k$$

At the pt $(1,1,1)$ $\frac{d\bar{r}}{dt} = i + 2j + 3k$

Let $\bar{a} = i + 2j + 3k$.

$x = t$, $y = t^2$, $z = t^3$

We have pt $(1,1,1)$

At $x = 1$, $t = 1$

At $y = 1$, $t = 1$

At $z = 1$, $t = 1$

$\therefore$ The t value is 1

$t = 1$.

Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (3i+3j+3k) \cdot \frac{(i+2j+3k)}{\sqrt{1+4+9}} = \frac{3+6+9}{\sqrt{14}}$$

$$= \frac{18}{\sqrt{14}}$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{3^2+3^2+3^2} = \sqrt{27} = 3\sqrt{3}$.

2) Find the directional derivative of $x^2y^2+y^2z^2+z^2x^2$ at the point $(1,1,-2)$ in the direction of the tangent to the curve $x = e^{-t}$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$. Also find maximum directional derivative. $\left[\text{Ans:- } \frac{2}{\sqrt{6}}\right]$

Sol:- The directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = x^2y^2+y^2z^2+z^2x^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 2xy^2+2xz^2 \quad \frac{\partial\phi}{\partial y} = 2yz^2+2yx^2 \quad \frac{\partial\phi}{\partial z} = 2zy^2+2zx^2$$

At the point $(1,1,-2)$. $\frac{\partial\phi}{\partial x} = 10 \quad \frac{\partial\phi}{\partial y} = 10, \quad \frac{\partial\phi}{\partial z} = -8$

$$\nabla\phi = 10i+10j-8k.$$

Given that the curve $x = e^{-t}$ $y = 2\sin t+1$ $z = t-\cos t$ —— (1)

Let $\bar{r}$ be the position vector of any point on the curve (1)

$$\bar{r} = xi+yj+zk$$

$$\bar{r} = e^{-t}i + (2\sin t+1)j + (t-\cos t)k$$

$$\frac{d\bar{r}}{dt} = -e^{-t}i + 2\cos t\, j + (1+\sin t)k$$

At $t=0$, $\frac{d\bar{r}}{dt} = -i+2j+k$

Let $\bar{a} = -i+2j+k$.

Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (10i+10j-8k) \cdot \frac{(-i+2j+k)}{\sqrt{1+4+1}}$$

$$= \frac{-10+20-8}{\sqrt{6}}$$

$$= \frac{2}{\sqrt{6}}$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{10^2+10^2+(-8)^2}$

$$= \sqrt{264}$$

TYPE-4 :-

1) Find the directional derivative of xyz^2+xz at $(1,1,1)$ in a direction of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$.

Sol: WKT The directional derivative of ϕ in the direction of $\bar{a}$ is given by $\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

Let $\phi = xyz^2+xz$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz^2+z \qquad \frac{\partial\phi}{\partial y} = xz^2 \qquad \frac{\partial\phi}{\partial z} = 2xyz+x$$

At the pt $(1,1,1)$ $\quad \frac{\partial\phi}{\partial x} = 2 \quad \frac{\partial\phi}{\partial y} = 1 \quad \frac{\partial\phi}{\partial z} = 3$.

$$\nabla\phi = 2i+j+3k.$$

Let $f = 3xy^2+y-z$

Normal to the surface f is given by $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = 3y^2 \qquad \frac{\partial f}{\partial y} = 6xy+1 \qquad \frac{\partial f}{\partial z} = -1.$$

At the pt $(0,1,1)$ $\quad \frac{\partial f}{\partial x} = 3 \quad \frac{\partial f}{\partial y} = 1 \quad \frac{\partial f}{\partial z} = -1$

$$\nabla f = 3i+j-k$$

Let $\bar{a} = 3i+j-k$

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (2i+j+3k) \cdot \frac{(3i+j-k)}{\sqrt{9+1+1}} = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

$\therefore$ Maximum Directional Derivative $= |\nabla\phi| = \sqrt{4+1+9} = \sqrt{14}$.

(2) Find the directional derivative of $x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the normal to the surface $x\log z - y^2$ at $(-1,2,1)$. Also find maximum directional derivative. Ans:- $\frac{14}{\sqrt{17}}$

Sol:- The directional derivative of ϕ in the direction of $\bar{a}$ is given by

$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

Let $\phi = x^2yz + 4xz^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2 \quad \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz.$$

At the point $(1,-2,-1)$ $\frac{\partial\phi}{\partial x} = 8$ $\frac{\partial\phi}{\partial y} = -1$ $\frac{\partial\phi}{\partial z} = -10$.

$$\nabla\phi = 8i - j - 10k.$$

Let $f = x\log z - y^2$.

Normal to the surface f is given by $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = \log z \quad \frac{\partial f}{\partial y} = -2y \quad \frac{\partial f}{\partial z} = \frac{x}{z}$$

At the point $(-1,2,1)$ $\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = -4$ $\frac{\partial f}{\partial z} = -1$.

$$\nabla f = -4j - k.$$

Let $\bar{a} = -4j - k$.

$\therefore$ Directional Derivative $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= (8i - j - 10k) \cdot \frac{(-4j - k)}{\sqrt{16+1}}$$

$$= \frac{4+10}{\sqrt{17}}$$

$$= \frac{14}{\sqrt{17}}$$

Maximum Directional Derivative $= |\nabla\phi| = \sqrt{8^2 + (-1)^2 + (-10)^2} = \sqrt{165}$.

TYPE-5 :-

1) In what direction from the point $(-1,1,2)$ is the directional derivative of xy^2z^3 maximum. What is the magnitude of the maximum.

Sol: Let $\phi = xy^2z^3$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y^2z^3 \quad \frac{\partial\phi}{\partial y} = 2xyz^3 \quad \frac{\partial\phi}{\partial z} = 3xy^2z^2$$

At the pt $(-1,1,2)$ $\quad \frac{\partial\phi}{\partial x} = 8 \quad \frac{\partial\phi}{\partial y} = -16 \quad \frac{\partial\phi}{\partial z} = -12$

$$\nabla\phi = 8i - 16j - 12k.$$

WKT the directional derivative of ϕ is maximum in the direction of $\nabla\phi$.

$\therefore$ The directional derivative is maximum in the direction of $\nabla\phi$.

$\therefore$ The magnitude of this maximum is $|\nabla\phi| = \sqrt{8^2 + (-16)^2 + (-12)^2}$

$$= \sqrt{464}$$

2) Find the maximum value of the directional derivative of $\phi = x^2yz$ at $(1,4,1)$

[Ans:- 9]

Sol:- Let $\phi = x^2yz$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 2xyz \quad \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y.$$

At the point $(1,4,1)$ $\quad \frac{\partial\phi}{\partial x} = 8 \quad \frac{\partial\phi}{\partial y} = 1 \quad \frac{\partial\phi}{\partial z} = 4$

$$\nabla\phi = 8i + j + 4k.$$

WKT the directional derivative of ϕ is maximum in the direction of $\nabla\phi$.

$\therefore$ The directional derivative is maximum in the direction of $\nabla\phi$.

$\therefore$ The magnitude of this maximum is $|\nabla\phi| = \sqrt{8^2 + 1^2 + 4^2}$

$$= \sqrt{69}.$$

TYPE - 6

→ Find the directional derivative of $5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$.

Sol:- We know that the directional derivative of the function ϕ in the direction of $\bar{a}$ is given by $\nabla\phi.\frac{\bar{a}}{|\bar{a}|}$

Let $\phi = 5x^2y - 5y^2z + 2.5z^2x$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$\frac{\partial\phi}{\partial x} = 10xy + 2.5z^2$ $\quad \frac{\partial\phi}{\partial y} = -10yz + 5x^2$ $\quad \frac{\partial\phi}{\partial z} = -5y^2 + 5zx.$

At the point $(1,1,1)$ $\frac{\partial x}{\partial x} = 12.5$ $\quad \frac{\partial\phi}{\partial y} = -5$ $\quad \frac{\partial\phi}{\partial z} = 0.$

$$\therefore \nabla\phi = 12.5i - 5j + 0.k.$$

Given that $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z-0}{1}.$

The direction of given line is $2i - 2j + 1k = \bar{a}.$

$\therefore$ Directional Derivative $= \nabla\phi.\frac{\bar{a}}{|\bar{a}|}$

$$= (12.5i - 5j + 0k).\frac{2i-2j+1k}{\sqrt{4+4+1}}$$

$$= \frac{25+10}{3} = \frac{35}{3}.$$

Find the values of constants a, b, c so that the directional derivative of $f = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has max. magnitude 64 in the direction parallel to the z-axis.

Sol:- Given that $f = axy^2 + byz + cz^2x^3$.

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}.$$

$$\nabla f = i(ay^2 + 3cx^2z^2) + j(2axy + bz) + k(by + 2czx^3)$$

At the point $(1, 2, -1)$ ——(1)

$$\nabla f = i(4a + 3c) + j(4a - b) + k(2b - 2c)$$

Given that Max. magnitude $= 64$ i.e $|\nabla f| = 64$.

$$\Rightarrow \sqrt{(4a+3c)^2 + (4a-b)^2 + (2b-2c)^2} = 64$$

$$(4a+3c)^2 + (4a-b)^2 + (2b-2c)^2 = 64^2 \quad \text{——(2)}$$

We find directional derivative of f in the direction of parallel to z-axis. i.e perpendicular to x-axis and y-axis.

Along x-axis unit vector is $\bar{a} = i$

$\therefore \nabla f.\bar{a} = 0 \Rightarrow \nabla f.\bar{a} = \nabla f.i = 4a + 3c = 0$ ——(3)

Along y-axis unit vector is $\bar{a} = j$

$\nabla f.\bar{a} = 0 \Rightarrow \nabla f.\bar{a} = 4a - b = 0$ ——(4)

Sub. (3) and (4) in (2), we get

$$b - c = 32 \quad \text{——(5)}$$

Solving (3), (4) and (5), we get.

$$\therefore a = 6$$
$$b = 24$$
$$c = -8$$

Divergence of a vector :-

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial \bar{f}}{\partial x}+\bar{j}.\frac{\partial \bar{f}}{\partial y}+\bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as $\operatorname{div}\bar{f}$

i.e $\operatorname{div}\bar{f}=\bar{i}.\frac{\partial \bar{f}}{\partial x}+\bar{j}.\frac{\partial \bar{f}}{\partial y}+\bar{k}.\frac{\partial \bar{f}}{\partial z}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\bar{f}$

Hence we can write $\operatorname{div}\bar{f}$ as $\operatorname{div}\bar{f}=\nabla.\bar{f}$

This is a scalar point function.

Note :-

(i) If the vector function $\bar{f}=f_1 i+f_2 j+f_3 k$ then $\operatorname{div}\bar{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

(ii) If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x},\frac{\partial f_2}{\partial y},\frac{\partial f_3}{\partial z}$ are zeros.

$\therefore \operatorname{div}\bar{f}=0$ for a constant vector $\bar{f}$

(iii) $\operatorname{div}(\bar{f}\pm\bar{g})=\operatorname{div}\bar{f}\pm\operatorname{div}\bar{g}$

(iv) $\nabla.\bar{f}\neq\bar{f}.\nabla$

Solenoidal vector :-

A vector point function $\bar{f}$ is said to be solenoidal if $\operatorname{div}\bar{f}=0$.

This equation is also called the equation of continuity or conservation of mass.

1) If $\bar{f}=(x+3y)i+(y-2z)j+(x+pz)k$ is solenoidal, find p.

Sol:- Let $\bar{f}=(x+3y)i+(y-2z)j+(x+pz)k$.

Let $\bar{f}=f_1 i+f_2 j+f_3 k$.

$f_1=x+3y \quad f_2=y-2z \quad f_3=x+pz$

$\frac{\partial f_1}{\partial x}=1 \quad \frac{\partial f_2}{\partial y}=1 \quad \frac{\partial f_3}{\partial z}=p$

$\operatorname{div}\bar{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=1+1+p=2+p$

Since $\bar{f}$ is solenoidal we have $\operatorname{div}\bar{f}=0 \implies 2+p=0$

$p=-2$

2) Find div.$\bar{f}$ when $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

sol:- Let $\phi = x^3+y^3+z^3-3xyz$.

$$\text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 3x^2-3yz \qquad \frac{\partial\phi}{\partial y} = 3y^2-3xz \qquad \frac{\partial\phi}{\partial z} = 3z^2-3xy.$$

$$\bar{f} = \text{grad}\,\phi = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k = f_1 i + f_2 j + f_3 k.$$

$$\text{div}\bar{f} = i\frac{\partial f_1}{\partial x} + j\frac{\partial f_2}{\partial y} + k\frac{\partial f_3}{\partial z}.$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x+6y+6z.$$

(3) If $f = (x^2+y^2+z^2)^{-n}$ then find divgrad$f$ and determine n if div grad$f = 0$.

sol:- Given that $f = (x^2+y^2+z^2)^{-n}$.

WKT $\text{grad}f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial x} = -2xn(x^2+y^2+z^2)^{-n-1} \quad \frac{\partial f}{\partial y} = -2yn(x^2+y^2+z^2)^{-n-1} \quad \frac{\partial f}{\partial z} = -2zn(x^2+y^2+z^2)^{-n-1}$$

$$\text{grad}f = (x^2+y^2+z^2)^{-n-1}(-2xni - 2ynj - 2znk)$$

$$\text{grad}f = -2n(x^2+y^2+z^2)^{-n-1}(xi+jy+kz)$$

WKT $\text{div}f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.

Let $f_1 = -2nx(x^2+y^2+z^2)^{-n-1}$ $f_2 = -2ny(x^2+y^2+z^2)^{-n-1}$ $f_3 = -2nz(x^2+y^2+z^2)^{-n-1}$

$$\frac{\partial f_1}{\partial x} = -2n(x^2+y^2+z^2)^{-n-1} + 2n(n+1).2x^2(x^2+y^2+z^2)^{-n-2}$$

$$\frac{\partial f_2}{\partial y} = -2n(x^2+y^2+z^2)^{-n-1} + 4n(n+1)y^2(x^2+y^2+z^2)^{-n-2}$$

$$\frac{\partial f_3}{\partial z} = -2n(x^2+y^2+z^2)^{-n-1} + 4n(n+1)z^2(x^2+y^2+z^2)^{-n-2}$$

$$\text{divgrad}f = -2n(x^2+y^2+z^2)^{-n-1}\left[3 - 2(n+1)(x^2+y^2+z^2)(x^2+y^2+z^2)^{-1}\right]$$

$$\text{divgrad}f = -2n(x^2+y^2+z^2)^{-n-1}(-2n+1).$$

$$= 2n(2n-1)$$

divgrad$f = 0$ when $n=0$, $n = \frac{1}{2}$

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Find div $\bar{F}$ where $\bar{F} = r^n \bar{r}$ Find n if it is solenoidal. [OR]

Prove that $r^n \bar{r}$ is solenoidal if $n = -3$

Sol:- Given that $\bar{F} = r^n \bar{r}$ Where $\bar{r} = xi + yj + zk$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$\bar{F} = (x^2+y^2+z^2)^{n/2}(xi+yj+zk)$$

$$\bar{F} = x(x^2+y^2+z^2)^{n/2} \cdot i + y(x^2+y^2+z^2)^{n/2} j + z(x^2+y^2+z^2)^{n/2} k.$$

Let $\bar{F} = f_1 i + f_2 j + f_3 k$.

Here $f_1 = x(x^2+y^2+z^2)^{n/2}$ $f_2 = y(x^2+y^2+z^2)^{n/2}$ $f_3 = z(x^2+y^2+z^2)^{n/2}$

We have $\operatorname{div}\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{n/2}\right]$$

$$= 1 \cdot (x^2+y^2+z^2)^{n/2} + x \cdot \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot (2x)$$

$$= (x^2+y^2+z^2)^{n/2} + n x^2 (x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= (r^2)^{n/2} + n x^2 (r^2)^{\frac{n-2}{2}} \quad \left[\because x^2+y^2+z^2 = r^2\right]$$

$$\frac{\partial f_1}{\partial x} = r^n + n x^2 r^{n-2}$$

Similarly $\frac{\partial f_2}{\partial y} = r^n + n y^2 r^{n-2}$ $\frac{\partial f_3}{\partial z} = r^n + n z^2 r^{n-2}$

$$\operatorname{div}\bar{F} = (r^n + n x^2 r^{n-2}) + (r^n + n y^2 r^{n-2}) + (r^n + n z^2 r^{n-2})$$

$$= 3r^n + n r^{n-2}(x^2+y^2+z^2)$$

$$= 3r^n + n r^{n-2} \cdot r^2$$

$$\operatorname{div}\bar{f} = 3r^n + nr^n$$

$$\operatorname{div}\bar{f} = (3+n)r^n$$

Let $\bar{f} = r^n \bar{r}$ be solenoidal Then $\operatorname{div}\bar{f} = 0$.

$$\therefore \quad (n+3)r^n = 0$$

$$n = -3.$$

Curl of a vector :-

Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $i\times\frac{\partial \bar{f}}{\partial x}+j\times\frac{\partial \bar{f}}{\partial y}+k\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or $(\nabla\times\bar{f})$

$$\therefore \text{curl}\,\bar{f} = i\times\frac{\partial \bar{f}}{\partial x}+j\times\frac{\partial \bar{f}}{\partial y}+k\times\frac{\partial \bar{f}}{\partial z}.$$

[OR]

If $\bar{f}$ is a differentiable vector function then curl $\bar{f}$ is defined as.

curl $\bar{f} = \nabla\times\bar{f}$.

If $\bar{f} = f_1 i+f_2 j+f_3 k$ then $\text{curl}\,\bar{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

$$\text{curl}\,\bar{f} = i\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-j\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)+k\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right).$$

Note:-

(i) The curl of a vector is also a vector.

(ii) If $\bar{f}$ is a constant vector then curl $\bar{f}=\bar{0}$.

(iii) $\text{curl}(\bar{f}\pm\bar{g}) = \text{curl}\,\bar{f}\pm\text{curl}\,\bar{g}$.

Irrotational Motion, Irrotational Vector :-

Any motion in which curl of the vector velocity is a null vector i.e curl $\bar{v}=\bar{0}$ is said to be irrotational.

→ A vector $\bar{f}$ is said to be irrotational if curl $\bar{f}=\bar{0}$

→ If $\bar{f}$ is irrotational, there will always exists a scalar function $\phi(x,y,z)$ such that $\bar{f}=\text{grad}\,\phi$. This ϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}=\text{grad}\,\phi$, then curl $\bar{f}=\bar{0}$.

Hence $\nabla\times\bar{f}=\bar{0}\Longleftrightarrow$ there exists a scalar function ϕ such that $\bar{f}=\nabla\phi$.

This idea is useful when we study the "work done by a force".

1) If $\bar{F} = xy\,i + yz\,j + zx\,k$ find curl $\bar{F}$ at $(1, 2, -3)$.

sol:- Given that $\bar{F} = xy\,i + yz\,j + zx\,k$. i.e $\bar{F} = f_1 i + f_2 j + f_3 k$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Here $f_1 = xy$

$f_2 = yz$

$f_3 = zx$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k. \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz)\right] - j\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial z}(xy)\right] + k\left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right]$$

$$= i(0-y) - j(z-0) + k(0-x)$$

$$\text{curl}\,\bar{F} = -yi - zj - xk$$

At the point $(1,2,-3)$ curl $\bar{F} = -2i + 3j - k$.

2) Find curl $\bar{F}$ where $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$ (or) $\bar{F} = \nabla(x^3+y^3+z^3-3xyz)$

sol:- Let $\phi = x^3+y^3+z^3-3xyz$

$$\bar{F} = \text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\bar{F} = i(3x^2-3yz) + j(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

$$\text{curl}\,\bar{F} = \text{curl grad}\,\phi = \begin{vmatrix} i & j & k. \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i(-3x+3x) - j(-3y+3y) + k(-3z+3z)$$

$$= \bar{0}.$$

$\therefore$ curl grad $\phi = \bar{0}$

3) Find constants a, b, c so that the vector $\bar{F} = (x+2y+az)i+(bx-3y-z)j + (4x+cy+2z)k$ is irrotational. Also find ϕ such that $\bar{F} = \nabla\phi$.

Sol: Given that $\bar{F} = (x+2y+az)i+(bx-3y-z)j + (4x+cy+2z)k$.

If $\bar{F} = f_1 i + f_2 j + f_3 k$ then $\text{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

WKT If $\bar{F}$ is irrotational then $\text{curl}\bar{F} = \bar{0}$

Here $f_1 = x+2y+az$ $f_2 = bx-3y-z$ $f_3 = 4x+cy+2z$

$\text{curl}\bar{F} = \bar{0}$ i.e $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$.

$$i\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - j\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right] = 0i+0j+0k$$

$i(c+1) - j(4-a) + k(b-2) = 0i+0j+0k$.

$c+1 = 0 \Rightarrow c = -1$.

$a-4 = 0 \Rightarrow a = 4$

$b-2 = 0 \Rightarrow b = 2$.

$\therefore \bar{F} = (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k$.

Here $\bar{F}$ is irrotational.

If $\bar{F}$ is irrotational. then there exists a function ϕ such that $\bar{F} = \nabla\phi$.. ϕ is called scalar potential.

We have $\bar{F} = \nabla\phi$

i.e $\bar{F} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$(x+2y+4z)i+(2x-3y-z)j+(4x-y+2z)k = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$$

Comparing bothsides, we have.

$$\frac{\partial\phi}{\partial x} = x+2y+4z. \text{ ——— } (1)$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \text{ ——— } (2)$$

$$\frac{\partial\phi}{\partial z} = 4x+y+2z \text{ ——— } (3).$$

Integrating (1) w.r.t 'x' partially (treating y and z as constant),

we get $\phi = \frac{x^2}{2}+2xy+4xz+c_1$ ——— (4)

Integrating (2) w.r.t 'y' partially (treating x and z as constant)

we get $\phi = 2xy-3\frac{y^2}{2}-yz+c_2$ ——— (5)

Integrating (3) w.r.t 'z' partially (treating x and y as constant)

we get $\phi = 4xz-yz+z^2+c_3$ ——— (6).

From (4), (5) and (6),

Hence $\phi = \frac{x^2}{2}-\frac{3y^2}{2}+z^2+2xy-yz+4xz+c$.

4) Show that the vector $(x^2-yz)i+(y^2-zx)j+(z^2-xy)k$ is irrotational and find its scalar potential.

Sol:- Let $\bar{F} = (x^2-yz)i+(y^2-zx)j+(z^2-xy)k = f_1 i + f_2 j + f_3 k$.

Here $f_1 = x^2-yz \quad f_2 = y^2-zx \quad f_3 = z^2-xy$

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-zx)\right] - j\left[\frac{\partial}{\partial x}(z^2-xy) - \frac{\partial}{\partial z}(x^2-yz)\right] + k\left[\frac{\partial}{\partial x}(y^2-zx) - \frac{\partial}{\partial y}(x^2-yz)\right]$$

$$= i(-x+x) - j(-y+y) + k(-z+z)$$

$\text{Curl}\,\bar{F} = \bar{0}$

$\therefore$ f is irrotational.

If $\bar{F}$ is irrotational then there exists a function ϕ such that $\bar{F} = \nabla\phi$. ϕ is called scalar potential.

We have $\bar{F} = \nabla\phi$ i.e $\bar{F} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$(x^2-yz)i+(y^2-zx)j+(z^2-xy)k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Comparing components bothsides, we get

$$\frac{\partial\phi}{\partial x} = x^2-yz \quad \text{—} (1)$$

$$\frac{\partial\phi}{\partial y} = y^2-zx \quad \text{—} (2)$$

$$\frac{\partial \phi}{\partial z} = z^2 - xy \quad \text{--- (3)}$$

Integrating (1) w.r.t x partially (treating y and z as constant)

we get $\phi = \int (x^2 - yz)\, dx + c_1$

$$\phi = \frac{x^3}{3} - xyz + c_1 \quad \text{--- (4)}$$

Integrating (2) w.r.t y partially (treating x and z as constant)

we get $\phi = \int (y^2 - zx)\, dy + c_2$

$$\phi = \frac{y^3}{3} - xyz + c_2 \quad \text{--- (5)}$$

Integrating (3) w.r.t z partially (treating x and y as constant)

we get $\phi = \int (z^2 - xy)\, dz$

$$\phi = \frac{z^3}{3} - xyz + c_3 \quad \text{--- (6)}$$

From (4) (5) and (6).

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c.$$

which is the required scalar potential ϕ.

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5) If $\bar{f} = (4x+3y+az)i + (bx-y+z)j + (2x+cy+z)k$ is irrotational. Find the constants a, b, c. Also find ϕ such that $\bar{f} = \nabla\phi$.

Sol:- Given that $\bar{f} = (4x+3y+az)i + (bx-y+z)j + (2x+cy+z)k$

Given that $\bar{f}$ is irrotational then curl $\bar{f} = \bar{0}$; $\bar{f} = f_1 i + f_2 j + f_3 k$

i.e
$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4x+3y+az & bx-y+z & 2x+cy+z \end{vmatrix} = \bar{0}$$

$$i(c-1) - j(2-a) + k(b-3) = 0i + 0j + 0k.$$

Equating the corresponding components, we get

$c-1=0 \Rightarrow c=1$; $a-2=0 \Rightarrow a=2$; $b-3=0 \Rightarrow b=3$

$\therefore a=2, b=3, c=1.$

$\therefore \bar{f} = (4x+3y+2z)i + (3x-y+z)j + (2x+y+z)k$.

If $\bar{f}$ is irrotational then there exists a function ϕ such that $\bar{f} = \nabla\phi$.

ϕ is called scalar potential.

We have $\bar{f} = \nabla\phi$ i.e $\bar{f} = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$(4x+3y+2z)i + (3x-y+z)j + (2x+y+z)k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Comparing the corresponding components bothsides, we get

$$\frac{\partial\phi}{\partial x} = 4x+3y+2z \quad \text{——} (1)$$

$$\frac{\partial\phi}{\partial y} = 3x-y+z \quad \text{——} (2)$$

$$\frac{\partial\phi}{\partial z} = 2x+y+z \quad \text{——} (3)$$

Integrating (1) w.r.t 'x' partially (treating y and z as constant)

we get $\phi = \int (4x+3y+2z)\,dx + c_1$

$$\phi = 2x^2 + 3xy + 2xz + c_1 \quad \text{——} (4)$$

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Integrating (2) w.r.t 'y' partially (treating x and z as constant)

we get $\phi = \int (3x - y + z)\, dy + c_2$

$$\phi = 3xy - \frac{y^2}{2} + yz + c_2 \quad \text{—— (5)}$$

Integrating (3) w.r.t 'z' partially treating (x and y as constant)

we get $\phi = \int (2x + y + z)\, dz + c_3$

$$\phi = 2xz + yz + \frac{z^2}{2} + c_3 \quad \text{—— (6)}$$

From (4) (5) and (6)

$$\phi = 2x^2 - \frac{y^2}{2} + \frac{z^2}{2} + 3xy + 2xz + yz + c.$$

Which is the required scalar potential ϕ.

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Vector Integration

Indefinite Integral :-

Integration is the inverse operation of Differentiation.

Let $\bar{F}(t)$ be a differentiable vector function of a scalar variable t and let $\frac{d}{dt}\{\bar{F}(t)\} = \bar{f}(t)$. Then $\int \bar{f}(t)\,dt = \bar{F}(t)$ and $\bar{F}(t)$ is called the primitive of $\bar{f}(t)$. The set of all primitives of $\bar{f}(t)$, that is $\int \bar{f}(t)\,dt = \bar{F}(t)+\bar{c}$ where $\bar{c}$ is any arbitrary constant vector, is called indefinite integral of $\bar{f}(t)$.

Hence the indefinite integral of $\bar{f}(t)$ is not unique.

Properties :-

(i) $\int k\bar{f}(t)\,dt = k\int \bar{f}(t)\,dt$, k is a real constant.

(ii) $\int [\bar{f}(t) \pm \bar{g}(t)]\,dt = \int \bar{f}(t)\,dt \pm \int \bar{g}(t)\,dt$

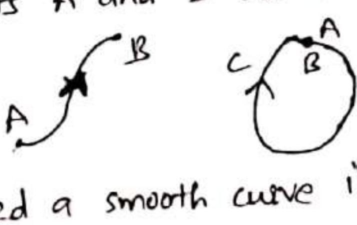
(iii) If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ then

$$\int \bar{f}(t)\,dt = \bar{i}\int f_1(t)\,dt + \bar{j}\int f_2(t)\,dt + \bar{k}\int f_3(t)\,dt + \bar{c}$$

Definite Integral :- Let $\int \bar{f}(t)\,dt = \bar{F}(t)+\bar{c}$ Then $\int_a^b \bar{f}(t)\,dt = \bar{F}(b)-\bar{F}(a)$.

This is called the Definite Integral of $\bar{f}(t)$ between the limits $t=a$ and $t=b$.

Closed Curve :- Let C be a curve in space. Let A be the initial point and B be the terminal point of the curve C. When the direction along C oriented from A to B is positive then the direction from B to A is called negative direction. If the two points A and B coincide the curve C is called the closed curve.



Smooth Curve :- A curve $\bar{r} = \bar{f}(t)$ is called a smooth curve if $\bar{f}(t)$ is continuously differentiable. A curve C is said to be piecewise smooth if it is the union of finite number of smooth curves.

Line Integrals :-

Let $\bar{F}(x,y,z)$ be a continuous vector function defined in the entire region of space. Let c be any curve in the region Divide c into n intervals by taking points $A = P_0, P_1, P_2, \ldots P_n = B$.

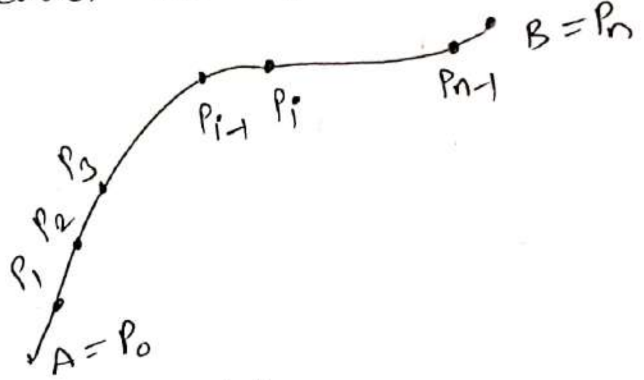
Let B_i be any point in the interval $P_{i-1}P_i$

Let $r_0, r_1, r_2, \ldots r_n$ be the position vectors of the points $P_0, P_1, P_2 \ldots P_n$ respectively. Let us consider the sum. $\Sigma \bar{F}(B_i)\, \delta r_i$

The limit of this sum as $n \to \infty$ and $|\delta r_i| \to 0$ is defined as the line integral of $\bar{F}$ along the curve c and is denoted symbolically by.

$\int_c \bar{F} \cdot d\bar{r}$ (or) $\int_c \bar{F} \cdot \frac{d\bar{r}}{dt}\, dt$ which is a scalar

If c is a closed curve, the integral is written as $\oint \bar{F} \cdot d\bar{r}$



Cartesian form of line integral :-

If $\bar{F} = f_1 i + f_2 j + f_3 k$, $d\bar{r} = dx\, i + dy\, j + dz\, k$

$$\int_c \bar{F} \cdot d\bar{r} = \int_c f_1\, dx + f_2\, dy + f_3\, dz.$$

Note :- $\int_c \phi\, d\bar{r}$ and $\int \bar{F} \times d\bar{r}$ are also examples of line integrals. In general any integral which is to be evaluated along a curve is called a line integral.

Circulation :- If $\bar{v}$ represents the velocity of a fluid particle and c is a closed curve, then the integral $\oint_c \bar{v}.d\bar{r}$ is called the circulation of $\bar{v}$ around the curve c.

If $\int_c \bar{v}.d\bar{r} = 0$ then the field $\bar{v}$ is called conservative. i.e no work is done. and the energy is conserved.

If the circulation of $\bar{v}$ around every closed curve in a region D vanishes then $\bar{v}$ is said to be irrotational in D. (i.e curl $\bar{v} = \bar{0}$)

Physical applications :-

If $\bar{F}$ represents the force vector acting on a particle moving along an arc AB, then the work done during a small displacement $\delta\bar{r}$ is $\bar{F}.\delta\bar{r}$. Hence the total work done by $\bar{F}$ during displacement from A to B is given by the line integral $\int_A^B \bar{F}.d\bar{r}$

If the force $\bar{F}$ is conservative i.e $\bar{F} = \nabla\phi$ then the work done is independent of the path and vice versa. In this case $\text{curl}\bar{F} = \text{curl}(\text{grad}\,\phi) = \bar{0}$ and ϕ is called scalar potential.

Note :- (i) $\bar{F}$ is conservative force field if $\nabla\times\bar{F} = \bar{0}$

(ii) A conservative force field is also irrotational i.e $\nabla\times\bar{F} = \bar{0}$

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TYPE-1

1) Using the line integral calculate the work done by the force $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ along the lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$.

Sol:- Given that $\bar{F} = (3x^2+6y)i - 14yz\,j + 20xz^2\bar{k}$

Given that the points $O(0,0,0)$ $A(1,0,0)$ $B(1,1,0)$ $C(1,1,1)$

We know that work done by the force $= \int_C \bar{F}\cdot d\bar{r}$

$$\text{Let } \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\,i + dy\,j + dz\,k$$

$$\bar{F}\cdot d\bar{r} = \left[(3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}\right]\cdot(dx\,i + dy\,j + dz\,k)$$

$$\bar{F}\cdot d\bar{r} = (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz \quad \text{--- (1)}$$

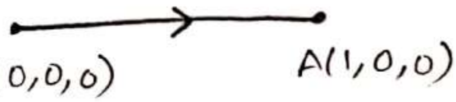
$$\text{Work } W = \int_C \bar{F}\cdot d\bar{r} = \int_{OA}\bar{F}\cdot d\bar{r} + \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} \quad \text{--- (2)}$$

Case (i) To evaluate $\int_{OA}\bar{F}\cdot d\bar{r}$ (or) Along the line OA :-

We have $O(0,0,0)$ $A(1,0,0)$.

Here $y=0$, $z=0$.

$dy=0$ $dz=0$.



x varies from 0 to 1.

$\therefore$ x limits are $x=0$, $x=1$

From (1), $\bar{F}\cdot d\bar{r} = 3x^2\,dx$.

$$\int_{OA}\bar{F}\cdot d\bar{r} = \int_{OA} 3x^2\,dx$$

$$= \int_{x=0}^{x=1} 3x^2\,dx = \left[3\frac{x^3}{3}\right]_{x=0}^{x=1}$$

$$= (1-0)$$

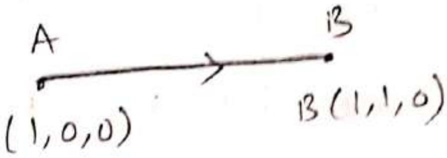
$$\int_{OA}\bar{F}\cdot d\bar{r} = 1$$

Case (ii) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(1,0,0)$ $B(1,1,0)$.

Here $x=1$, $z=0$

$dx=0$ $dz=0$



y varies from 0 to 1.

$\therefore$ y limits $y=0$, $y=1$.

From (1). $\bar{F}\cdot d\bar{r} = 0$

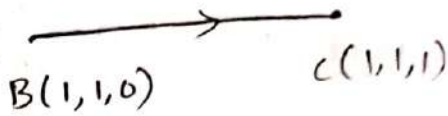
$$\int_{AB} \bar{F}\cdot d\bar{r} = 0 \quad \text{———} \ (4).$$

Case (iii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(1,1,0)$ $C(1,1,1)$

$x=1$ $y=1$

$dx=0$ $dy=0$



z varies from 0 to 1.

$\therefore$ z limits $z=0$, $z=1$.

From (1), $\bar{F}\cdot d\bar{r} = 20z^2\,dz$.

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} 20z^2\,dz$$

$$= \int_{z=0}^{z=1} 20z^2\,dz$$

$$= \left[20\frac{z^3}{3}\right]_{z=0}^{z=1}$$

$$= \frac{20}{3} \quad \text{———} \ (5)$$

Sub. (3), (4) and (5) in (2), we get

$$\int_C \bar{F}\cdot d\bar{r} = 1+0+\frac{20}{3} = \frac{23}{3}$$

2 If $\bar{F} = (x^2-27)i - 6yz\,j + 8xz^2\,k$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the straight line from $(0,0,0)$ to $(1,0,0)$, $(1,0,0)$ to $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$.

Ans: $-\frac{72}{3}$

Sol:- Given that $\bar{F} = (x^2-27)i - 6yz\,j + 8xz^2\,k$.

Given that the points $O(0,0,0)$ $A(1,0,0)$ $B(1,1,0)$ $C(1,1,1)$.

We have to find $\int_c \bar{F}\cdot d\bar{r}$.

$$\text{Let } \bar{r} = xi + yj + zk.$$

$$d\bar{r} = dx\,i + dy\,j + dz\,k.$$

$$\bar{F}\cdot d\bar{r} = \left[(x^2-27)i - 6yz\,j + 8xz^2\,k\right]\cdot\left[dx\,i + dy\,j + dz\,k\right]$$

$$\bar{F}\cdot d\bar{r} = (x^2-27)\,dx - 6yz\,dy + 8xz^2\,dz \quad \text{—— (1)}$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} \quad \text{—— (2)}$$

Case(i) To evaluate $\int_{OA} \bar{F}\cdot d\bar{r}$ (or) Along the line OA :-

We have $O(0,0,0)$ $A(1,0,0)$

Here $y=0$ $z=0$

$dy=0$ $dz=0$

x varies from 0 to 1.

$\therefore$ x limits $x=0$, $x=1$.

From (1), $\bar{F}\cdot d\bar{r} = (x^2-27)\,dx$.

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{OA} (x^2-27)\,dx$$

$$= \int_{x=0}^{x=1} (x^2-27)\,dx = \left[\frac{x^3}{3} - 27x\right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{3} - 27\right) - 0$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = -\frac{80}{3} \quad \text{—— (3)}$$

Case(ii) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(1,0,0)$ $B(1,1,0)$

Here $x=1$, $z=0$

$dx=0$ $dz=0$

y varies from 0 to 1

$\therefore$ y limits $y=0$, $y=1$

From (1) $\bar{F}\cdot d\bar{r} = 0$

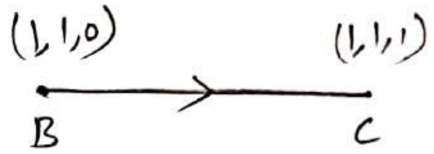
$$\int_{AB} \bar{F}\cdot d\bar{r} = 0 \quad \text{--- (4)}$$

Case(iii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(1,1,0)$ $C(1,1,1)$

Here $x=1$, $y=1$

$dx=0$ $dy=0$



z varies from 0 to 1

$\therefore$ z limits $z=0$, $z=1$

From (1) $\bar{F}\cdot d\bar{r} = 8z^2\,dz$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} 8z^2\,dz$$

$$= \int_{z=0}^{z=1} 8z^2\,dz$$

$$= \left[\frac{8z^3}{3}\right]_{z=0}^{z=1}$$

$$= \frac{8}{3} \quad \text{--- (5)}$$

sub. (3) (4) and (5) in (2), we get

$$\int_C \bar{F}\cdot d\bar{r} = -\frac{80}{3} + \frac{8}{3}$$

$$\int_C \bar{F}\cdot d\bar{r} = -\frac{72}{3}$$

TYPE - 2

→ Find the work done by the force $\bar{F} = (3x^2 - 6yz)i + (2y + 3xz)j + (1 - 4xyz^2)k$ in moving particle from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve C : $x = t$ $y = t^2$ $z = t^3$.

Sol: Given that $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y + 3xz)\bar{j} + (1 - 4xyz^2)\bar{k}$

WKT Work done by the force $= \int_C \bar{F} \cdot d\bar{r}$

Let $\bar{r} = xi + yj + zk$.

Given that the curve C is $x = t$ $y = t^2$ $z = t^3$

$dx = dt$ $dy = 2t\,dt$ $dz = 3t^2\,dt$

$$d\bar{r} = dx\,i + dy\,j + dz\,k.$$

$$d\bar{r} = dt\,i + 2t\,dt\,j + 3t^2\,dt\,k.$$

$$\bar{F} = (3t^2 - 6t^5)i + (2t^2 + 3t^4)j + (1 - 4t^9)k.$$

$$\bar{F} \cdot d\bar{r} = [(3t^2 - 6t^5)i + (2t^2 + 3t^4)j + (1 - 4t^9)k] \cdot [dt\,i + 2t\,dt\,j + 3t^2\,dt\,k]$$

$$= (3t^2 - 6t^5)dt + (2t^2 + 3t^4)\,2t\,dt + (1 - 4t^9)\,3t^2\,dt.$$

$$\bar{F} \cdot d\bar{r} = (-12t^{11} + 4t^3 + 6t^2)\,dt$$

The end points are $O(0,0,0)$ $A(1,1,1)$

We have $x = t$ $y = t^2$ $z = t^3$

At $x = 0$, $t = 0$ At $x = 1$, $t = 1$.

At $y = 0$, $t = 0$ At $y = 1$, $t = 1$

At $z = 0$, $t = 0$ At $z = 1$, $t = 1$

∴ The corresponding values of t are $t = 0$ and $t = 1$.

∴ t limits are $t = 0$, $t = 1$

$$\therefore \text{Work} = \int_C \bar{F} \cdot d\bar{r}$$

$$= \int_C (-12t^{11} + 4t^3 + 6t^2)\,dt = \int_{t=0}^{t=1} (-12t^{11} + 4t^3 + 6t^2)\,dt$$

$$= \left[-t^{12} + t^4 + 2t^3\right]_{t=0}^{t=1}$$

Work. $= 2$

2. If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$. Evaluate $\int_c \bar{F} \cdot d\bar{r}$. Ans:- $\frac{51}{70}$.

Sol:- Given that $\bar{F} = xy\, i - z\, j + x^2 k$.

Given that the curve $x = t^2$ $y = 2t$ $z = t^3$.

We have to find $\int_c \bar{F} \cdot d\bar{r}$

$$r = xi + yj + zk.$$

$$d\bar{r} = dx\, i + dy\, j + dz\, k.$$

$$dx = 2t\,dt \quad dy = 2dt \quad dz = 3t^2\, dt$$

$$d\bar{r} = (2t\, i + 2j + 3t^2 k)\, dt$$

$$\bar{F} = 2t^3\, i - t^3\, j + t^4 k.$$

$$\bar{F} \cdot d\bar{r} = (2t^3 i - t^3 j + t^4 k) \cdot (2t\,dt\, i + 2dt\, j + 3t^2 dt\, k)$$

$$\bar{F} \cdot d\bar{r} = (4t^4 - 2t^3 + 3t^6)\, dt$$

Given that t varies from 0 to 1.

$$\int_c \bar{F} \cdot d\bar{r} = \int_c (4t^4 - 2t^3 + 3t^6)\, dt$$

$$= \int_{t=0}^{t=1} (4t^4 - 2t^3 + 3t^6)\, dt$$

$$= \left[\frac{4t^5}{5} - \frac{2t^4}{4} + \frac{3t^7}{7}\right]_{t=0}^{t=1}$$

$$= \left(\frac{4}{5} - \frac{2}{4} + \frac{3}{7}\right) - 0$$

$$= \frac{112 - 70 + 60}{140}$$

$$\int_c \bar{F} \cdot d\bar{r} = \frac{51}{70}$$

Note:- Let two points be (x_1, y_1, z_1) (x_2, y_2, z_2). Then the equation of the line $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ (symmetric form)

TYPE-3

1) Find the work done in moving a particle in the force field $\bar{F} = 3x^2 i + j + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

sol:- Given that $\bar{F} = 3x^2 i + j + zk$

WKT the Work done by the force $= \int_c \bar{F}.d\bar{r}$

$$\text{Let } \bar{r} = xi + yj + zk.$$

$$d\bar{r} = dx\, i + dy\, j + dz\, k.$$

Given that $O(0,0,0)$ $A(2,1,3)$

Equation of OA is $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$.

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$$x = 2t \quad y = t \quad z = 3t$$

$$dx = 2dt \quad dy = dt \quad dz = 3dt$$

$$d\bar{r} = 2dt\, i + dt\, j + 3\, dt\, k.$$

The points $(0,0,0)$ and $(2,1,3)$ Correspond to $t=0$ and $t=1$ respectively

We have $O(0,0,0)$ $A(2,1,3)$

when $x=0, y=0\ z=0$.
$\Rightarrow t=0$.
$\therefore x = 2t, y = t, z = 3t$

when $x=2, y=1, z=3$
$\Rightarrow t=1$.

$\therefore$ t limits are $t=0, t=1$.

$$\bar{F}.d\bar{r} = (3x^2 i + j + zk).(2dt\, i + dt\, j + 3dt\, k)$$

$$= 3x^2(2dt) + dt + z(3dt)$$

$$= 3(2t)^2 2dt + dt + (3t)(3dt)$$

$$\bar{F}.d\bar{r} = 24t^2 dt + dt + 9t\, dt.$$

$$\int_c \bar{F}.d\bar{r} = \int_c 24t^2 dt + 9t\, dt + dt$$

$$= \int_{t=0}^{t=1} (24t^2 dt + 9t\, dt + dt) = \left[24.\frac{t^3}{3} + 9\frac{t^2}{2} + t\right]_{t=0}^{t=1}$$

$$= 8 + \frac{9}{2} + 1 = \frac{27}{2}$$

→ Find the work done in moving a particle in the force field $\bar{F} = 3x^2 i + (2xz - y) j + zk$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Ans:- 16.

Sol:- Given that $\bar{F} = 3x^2 i + (2xz - y) j + zk$.

Wkt Work done by the force. $= \int_c \bar{F} \cdot d\bar{r}$

Let $\bar{r} = xi + yj + zk$.

$d\bar{r} = dx\, i + dy\, j + dz\, k$

Given that $O(0,0,0)$ $A(2,1,3)$

Equation of OA is. $\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$.

$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say)

$x = 2t \quad y = t \quad z = 3t$

$dx = 2\,dt \quad dy = dt \quad dz = 3\,dt$

$d\bar{r} = (2i + j + 3k)\,dt$

$\bar{F} = 12t^2 i + (12t^2 - t) j + 3t\, k$.

$\bar{F} \cdot d\bar{r} = [12t^2 i + (12t^2 - t) j + 3t\, k] \cdot (2i + j + 3k)\, dt$

$\bar{F} \cdot d\bar{r} = 24t^2 + 12t^2 - t + 9t = (36t^2 + 8t)\,dt$.

The points $O(0,0,0)$ $A(2,1,3)$ correspond to $t=0$ and $t=1$ respectively.

∴ t limits are $t=0, t=1$.

We have $x = 2t$, $y = t$, $z = 3t$

$O(0,0,0)$ $A(2,1,3)$

when $x=0, y=0, z=0$.

$\Rightarrow t = 0$.

when $x=2, y=1, z=3$.

$t=1$.

$$\int_c \bar{F} \cdot d\bar{r} = \int_c (36t^2 + 8t)\,dt$$

$$= \int_{t=0}^{t=1} (36t^2 + 8t)\,dt$$

$$= \left[36 \cdot \frac{t^3}{3} + 8 \cdot \frac{t^2}{2}\right]_{t=0}^{t=1}$$

$$= (12 + 4) - 0$$

$$\int_c \bar{F} \cdot d\bar{r} = 16$$

→ Prove that force field given by $\bar{F} = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$ is conservative. Find the work done by moving a particle from $(1,-1,2)$ to $(3,2,-1)$ in this force field.

Ans :- -10.

Sol:- If the integral $\int_{P_1}^{P_2} \bar{F}.d\bar{r}$ is independent of the path joining P_1 and P_2 then $\bar{F}$ is called a conservative field.

Alternatively, $\bar{F}$ is called a conservative field if curl $\bar{F} = \bar{0}$ iff there exists a scalar point function ϕ such that $\bar{F} = \nabla\phi$.

Given that $\bar{F} = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$ $\bar{F} = F_1 i + F_2 j + F_3 k$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$\text{curl}\,\bar{F} = i(3x^2z^2 - 3x^2z^2) - j(6xyz^2 - 6xyz^2) + k(2xz^3 - 2xz^3)$$

$$\text{curl}\,\bar{F} = \bar{0}$$

Hence $\bar{F}$ is conservative.

To find ϕ such that $\bar{F} = \nabla\phi$

$$2xyz^3 i + x^2z^3 j + 3x^2yz^2 k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \quad \frac{\partial\phi}{\partial y} = x^2z^3 \quad \frac{\partial\phi}{\partial z} = 3x^2yz^2$$

We have $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$d\phi = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$$

$$d\phi = d(x^2yz^3)$$

$$\Rightarrow \phi = x^2yz^3 + c.$$

Work done by the force $= \int_{P_1}^{P_2} \bar{F}.d\bar{r}$

We have $\bar{r} = xi + yj + zk \Rightarrow d\bar{r} = dx\,i + dy\,j + dz\,k$.

$$\bar{F}.d\bar{r} = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz = d(x^2yz^3)$$

$$\text{Hence Work} = \int_{P_1}^{P_2}\bar{F}\,d\bar{r} = \int_{(1,-1,2)}^{(3,2,-1)} d(x^2yz^3) = \left[x^2yz^3\right]_{(1,-1,2)}^{(3,2,-1)}$$

Work. $= -10$

TYPE-4

1) Find the circulation of $\bar{F} = (2x-y+2z)i + (x+y-z)j + (3x-2y-5z)k$ along the circle $x^2+y^2=4$ in the xy-plane.

Sol:- Given that $F = (2x-y+2z)i + (x+y-z)j + (3x-2y-5z)k$

WKT circulation $= \oint_c \bar{F}.d\bar{r}$ where c is the circle $x^2+y^2=4$

Let $\bar{r} = xi+yj+zk$.

$\bar{r} = xi+yj$ [$\because$ In xy-plane $z=0$]

$d\bar{r} = dx\,i + dy\,j + 0.k$.

$F = (2x-y)i + (x+y)j + (3x-2y)k$ [$\because$ In xy-plane $z=0$]

$F.d\bar{r} = [(2x-y)i + (x+y)j + (3x-2y)k].(dx\,i+dy\,j+0.k)$

$F.d\bar{r} = (2x-y)dx + (x+y)dy$

Given that the circle $x^2+y^2=4$.

$x = 2\cos\theta \quad y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta, \quad dy = 2\cos\theta\, d\theta$

In circle, θ varies from 0 to 2π

$\therefore$ θ limits $\theta=0, \theta=2\pi$

$F.d\bar{r} = (4\cos\theta - 2\sin\theta)(-2\sin\theta)d\theta + (2\cos\theta+2\sin\theta)(2\cos\theta)d\theta$

$F.d\bar{r} = [-8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta]d\theta$

$= (4-4\sin\theta\cos\theta)d\theta$

$F.d\bar{r} = (4-2\sin 2\theta)d\theta$

$\int_c F.d\bar{r} = \int_c (4-2\sin 2\theta)d\theta$

$= \int_{\theta=0}^{\theta=2\pi} (4-2\sin 2\theta)d\theta$

$= [4\theta + \cos 2\theta]_{\theta=0}^{\theta=2\pi} = 8\pi + \cos 4\pi - \cos 0.$

$= 8\pi$

2) Find the work done by $\bar{F} = (2x-y-z)i + (x+y-z)j + (3x-2y-5z)k$ along a curve C in the xy-plane given by $x^2+y^2=9$, $z=0$ Ans:- 18π

Sol:- Given that $\bar{F} = (2x-y-z)i + (x+y-z)j + (3x-2y-5z)k$.

Let $\bar{r} = xi + yj + zk$.

$d\bar{r} = dx\,i + dy\,j + dz\,k$.

In xy-plane. $z=0$ $\therefore dz = 0$.

$d\bar{r} = dx\,i + dy\,j$

$\bar{F} = (2x-y)i + (x+y)j + (3x-2y)k$. [$\because$ In xy-plane $z=0$]

$\bar{F}\cdot d\bar{r} = [(2x-y)i + (x+y)j + (3x-2y)k]\cdot[dx\,i + dy\,j + 0\cdot k]$

$\bar{F}\cdot d\bar{r} = (2x-y)dx + (x+y)dy$.

Given that the curve c in the xy-plane is $x^2+y^2=9$, $z=0$.

The parametric equations of circle $x^2+y^2=9$ are $x = 3\cos\theta$ $y = 3\sin\theta$

$dx = -3\sin\theta\, d\theta$ $dy = 3\cos\theta\, d\theta$

$\therefore \theta$ varies from 0 to 2π.

$\therefore F\cdot d\bar{r} = (6\cos\theta - 3\sin\theta)(-3\sin\theta)d\theta + (3\cos\theta + 3\sin\theta)(3\cos\theta)d\theta$

$F\cdot d\bar{r} = [-18\sin\theta\cos\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cos\theta]d\theta$

$F\cdot d\bar{r} = [9 - 9\sin\theta\cos\theta]d\theta$

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (9 - 9\sin\theta\cos\theta)d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} (9 - 9\sin\theta\cos\theta)d\theta$$

$$= 9[\theta]_{\theta=0}^{\theta=2\pi} - 9\left[\frac{\sin^2\theta}{2}\right]_{\theta=0}^{\theta=2\pi}$$

$$= 9(2\pi - 0) - \frac{9}{2}[\sin^2(2\pi) - \sin(0)]$$

$$\int_c \bar{F}\cdot d\bar{r} = 18\pi$$

→ A vector field is given by $\bar{F} = \sin y\, i + x(1+\cos y)\, j$. Evaluate the line integral over a circular path given by $x^2+y^2=a^2,\ z=0$.

Sol:- Given that $\bar{F} = \sin y\, i + x(1+\cos y)\, j$

We have $\bar{r} = xi + yj$

$d\bar{r} = dx\, i + dy\, j$

$$\bar{F}.d\bar{r} = [\sin y\, i + x(1+\cos y) j].[dx\, i + dy\, j]$$

$$\bar{F}.d\bar{r} = \sin y\, dx + x(1+\cos y)\, dy$$

Given that the circle $x^2+y^2=a^2$

The parametric equations are $x = a\cos\theta \quad y = a\sin\theta$

$dx = -a\sin\theta\, d\theta \quad dy = a\cos\theta\, d\theta$

θ varies from 0 to 2π

$\therefore \theta$ limits $\theta=0,\ \theta=2\pi$.

$$\oint_C \bar{F}.d\bar{r} = \oint_C \sin y\, dx + x(1+\cos y)\, dy.$$

$$= \int_{\theta=0}^{\theta=2\pi} \sin(a\sin\theta).(-a\sin\theta)d\theta + a\cos\theta(1+\cos(a\sin\theta))a\cos\theta d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} [-a\sin\theta\sin(a\sin\theta) + a^2\cos^2\theta + a^2\cos^2\theta\cos(a\sin\theta)]d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} a^2\cos^2\theta\, d\theta + \int_{\theta=0}^{\theta=2\pi} [a^2\cos^2\theta\cos(a\sin\theta) - a\sin\theta\sin(a\sin\theta)]d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} a^2\left(\frac{1+\cos 2\theta}{2}\right) + \int_{\theta=0}^{\theta=2\pi} d[a\cos\theta\sin(a\sin\theta)]$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi} + [a\cos\theta\sin(a\sin\theta)]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a^2}{2}\left[\left(2\pi + \frac{\sin 4\pi}{2}\right) - 0\right] + [a\cos(2\pi)\sin(a\sin(2\pi)) - 0]$$

$$\int_C \bar{F}.d\bar{r} = \pi a^2$$

$\longrightarrow$ If $\bar{F} = 2yi - zj + xk$, evaluate $\int_C \bar{F} \times d\bar{r}$ along the curve $x = \cos t$, $y = \sin t$

$z = 2\cos t$ from $t=0$ to $t = \pi/2$

Sol: Given that $\bar{F} = 2yi - zj + xk$. i.e $\bar{F} = F_1 i + F_2 j + F_3 k$

$\bar{r} = xi + yj + zk$. Here $F_1 = 2y$

$d\bar{r} = dx\, i + dy\, j + dz\, k$ $F_2 = -z$ $F_3 = x$.

We have to find $\int_C \bar{F} \times d\bar{r}$

$$\bar{F} \times d\bar{r} = \begin{vmatrix} i & j & k \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$\bar{F} \times d\bar{r} = i(-z\,dz - x\,dy) - j(2y\,dz - x\,dx) + k(2y\,dy + z\,dx) \quad \text{---} (1)$$

Given that the curve is. $x = \cos t$ $y = \sin t$ $z = 2\cos t$

$dx = -\sin t\, dt$ $dy = \cos t\, dt$ $dz = -2\sin t\, dt$ — (2)

sub. (2) in (1), we get

$$\bar{F} \times d\bar{r} = i(4\sin t\cos t - \cos^2 t)\,dt - j(-4\sin^2 t + \sin t\cos t)\,dt$$

$$\int_C \bar{F} \times d\bar{r} = \int_C [i(4\sin t\cos t - \cos^2 t)\,dt - j(\sin t\cos t - 4\sin^2 t)\,dt]$$

$$= i\int_{t=0}^{t=\pi/2} (4\sin t\cos t - \cos^2 t)\,dt + j\int_{t=0}^{t=\pi/2} (4\sin^2 t - \sin t\cos t)\,dt$$

$$= i\int_{t=0}^{t=\pi/2} \left(2\sin 2t - \left(\frac{1+\cos 2t}{2}\right)\right) dt + j\int_{t=0}^{t=\pi/2} \left(2(1-\cos 2t) - \frac{\sin 2t}{2}\right) dt$$

$$= i\left(2 - \frac{\pi}{4}\right) + j\left(\pi - \frac{1}{2}\right).$$

→ Evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ over arc of a helix $x = a\cos t$, $y = a\sin t$, $z = kt$ as t varies from 0 to 2π Ans:- 0.

Sol:- Given that $I = \oint_C (yz\,dx + xz\,dy + xy\,dz)$

Given that the curve is $x = a\cos t$ $y = a.\sin t$ $z = kt$.

$dx = -a\sin t\,dt$ $dy = a\cos t\,dt$ $dz = k\,dt$.

Given that t varies from 0 to 2π.

$$\oint_C (yz\,dx + xz\,dy + xy\,dz) = \oint_C akt\sin t(-a\sin t)dt + akt\cos t(a\cos t)\,dt + a^2\sin t\cos t(k\,dt)$$

$$= -a^2k\int_{t=0}^{t=2\pi} t\sin^2 t\,dt + a^2k\int_{t=0}^{t=2\pi} t\cos^2 t\,dt + a^2k\int_{t=0}^{t=2\pi}\sin t\cos t\,dt$$

$$= a^2k\int_{t=0}^{t=2\pi} t(\cos^2 t - \sin^2 t)\,dt + a^2k\int_{t=0}^{t=2\pi}\sin t\cos t\,dt$$

$$= a^2k\int_{t=0}^{t=2\pi} t\cos 2t\,dt + a^2k\left[\frac{\sin^2 t}{2}\right]_{t=0}^{t=2\pi}$$

$$= a^2k\left[t\left(\frac{\sin 2t}{2}\right) - 1.\left(\frac{-\cos 2t}{4}\right)\right]_{t=0}^{t=2\pi} + \frac{a^2k}{2}\left[\sin^2(2\pi) - \sin^2(0)\right]$$

$$= a^2k\left[\left\{2\pi\left(\frac{\sin 4\pi}{2}\right) + \frac{\cos 4\pi}{4}\right\} - \left(\frac{\cos 0}{4}\right)\right]$$

$$= 0$$

TYPE-5

→ If c is the curve $y = 3x^2$ in the xy-plane and $\bar{F} = (x+2y)i - xy\,j$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ from the point $(0,0)$ to $(1,3)$.

Sol:- Given that $\bar{F} = (x+2y)i - xy\,j$

We have to find $\int_c \bar{F}\cdot d\bar{r}$

We have $\bar{r} = xi + yj + zk$

$\bar{r} = xi + yj$ [$\because$ In xy-plane $z=0$]

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = [(x+2y)i - xy\,j]\cdot(dx\,i + dy\,j)$

$\bar{F}\cdot d\bar{r} = (x+2y)\,dx - xy\,dy$

Given that the curve $y = 3x^2$

$dy = 6x\,dx$.

Given that the points $O(0,0)$ $A(1,3)$

$\therefore$ x varies from 0 to 1.

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (x + 6x^2)\,dx - 3x^3 \cdot 6x\,dx.$$

$$= \int_c (x + 6x^2 - 18x^4)\,dx$$

$$= \int_{x=0}^{x=1} (x + 6x^2 - 18x^4)\,dx$$

$$= \left[\frac{x^2}{2} + 2x^3 - \frac{18}{5}x^5\right]_{x=0}^{x=1}$$

$$= \frac{-11}{10}.$$

→ If $\bar{F} = (5xy - 6x^2)i + (2y - 4x)j$ evaluate $\int_c \bar{F} \cdot d\bar{r}$ along the curve c In xy-plane $y = x^3$ from (1,1) to (2,8)

Ans:- 35

Sol:- Given that $\bar{F} = (5xy - 6x^2)i + (2y - 4x)j$

Given that the curve $y = x^3 \Rightarrow dy = 3x^2\,dx$.

$$\bar{F} = (5x^4 - 6x^2)i + (2x^3 - 4x)j \quad [\because y = x^3]$$

Let $\bar{r} = xi + yj$.

$$d\bar{r} = dx\,i + dy\,j \quad [\because \text{In } xy\text{-plane } z = 0]$$

$$d\bar{r} = dx\,i + 3x^2 dx\,j$$

$$\bar{F} \cdot d\bar{r} = [(5x^4 - 6x^2)i + (2x^3 - 4x)j] \cdot [dx\,i + 3x^2 dx\,j]$$

$$\bar{F} \cdot d\bar{r} = (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

We have A(1,1) B(2,8)

x varies from 1 to 2.

$\therefore$ x limits $x = 1, x = 2$.

$$\int_c \bar{F} \cdot d\bar{r} = \int_c (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx.$$

$$= \int_{x=1}^{x=2} (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

$$= \left[6 \cdot \frac{x^6}{6} + 5 \cdot \frac{x^5}{5} - 12\frac{x^4}{4} - 6 \cdot \frac{x^3}{3}\right]_{x=1}^{x=2}$$

$$= \left[x^6 + x^5 - 3x^4 - 2x^3\right]_{x=1}^{x=2}$$

$$= (2^6 + 2^5 - 3 \cdot 2^4 - 2 \cdot 2^3) - (1 + 1 - 3 - 2)$$

$$\int_c \bar{F} \cdot d\bar{r} = 35$$

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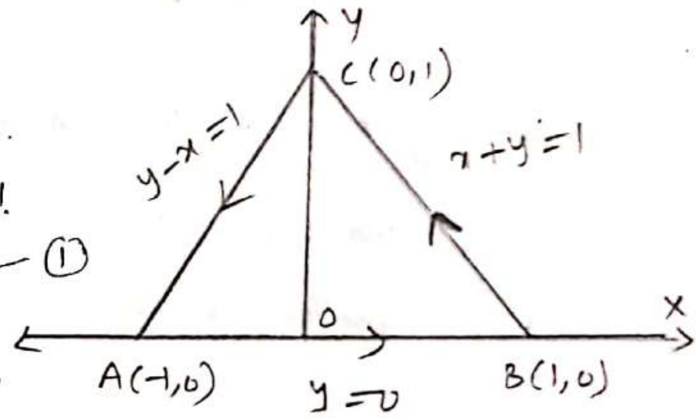
→ Compute the line integral $\int_C (y^2 dx - x^2 dy)$ round the triangle whose vertices are $(1,0)$ $(0,1)$ $(-1,0)$ in the xy-plane.

TYPE - 6

Sol: Let $I = \int_C y^2 dx - x^2 dy$ . .

Let the vertices of the triangle are $A(-1,0)$ $B(1,0)$ $C(0,1)$.



Equation of the line AB is $y=0$

Equation of the line BC is $x+y=1$.

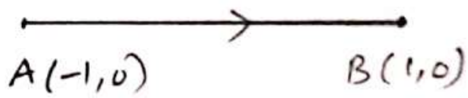
Equation of the line CA is $y-x=1$ — ①

$\therefore \int_C y^2 dx - x^2 dy = \int_{AB} + \int_{BC} + \int_{CA}$.

<u>Case (i)</u> :- Along the line AB (or) To evaluate $\int_{AB} y^2 dx - x^2 dy$.

We have $A(-1,0)$ $B(1,0)$

Here $y=0 \Rightarrow dy=0$.



x varies from -1 to 1.

$\therefore$ x limits are $x=-1, x=1$.

$$\int_{AB} y^2 dx - x^2 dy = \int_{AB} 0.dx - x^2.(0)$$

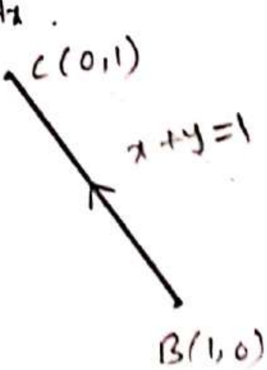
$$\int_{AB} y^2 dx - x^2 dy = 0 \quad \text{—— ②}$$

<u>Case (ii)</u> :- Along the line BC (or) To evaluate $\int_{BC} y^2 dx - x^2 dy$.

We have $B(1,0)$ $C(0,1)$

Equation of the line BC is $x+y=1 \Rightarrow y=1-x$

$dy = -dx$.



$\therefore$ x varies from 1 to 0

$\therefore$ x limits are $x=1, x=0$.

$$\int_{BC} y^2 dx - x^2 dy = \int_{BC} (1-x)^2 dx - x^2(-dx)$$

$$= \int_{x=1}^{x=0} \left[(x-1)^2 + x^2\right] dx$$

$$= \left[\frac{(x-1)^3}{3} + \frac{x^3}{3}\right]_{x=1}^{x=0}$$

$$= \left(-\frac{1}{3} + 0\right) - \left(0 + \frac{1}{3}\right)$$

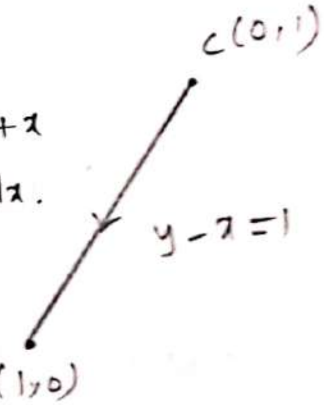
$$\int_{BC} y^2\,dx - x^2\,dy = -\frac{2}{3} \quad \text{——} (3)$$

<u>Case (iii)</u> Along the line CA (or) To evaluate $\int_{CA} y^2\,dx - x^2\,dy$.

We have $C(0,1)$ $A(-1,0)$

Equation of the line CA is $y - x = 1 \Rightarrow y = 1 + x$

$dy = dx$.



x varies from 0 to -1

$\therefore$ x limits $x = 0$, $x = -1$.

$$\int_{CA} y^2\,dx - x^2\,dy = \int_{CA} (x+1)^2\,dx - x^2\,dx.$$

$$= \int_{x=0}^{x=-1} \left[(x+1)^2 - x^2\right] dx$$

$$= \left[\frac{(x+1)^3}{3} - \frac{x^3}{3}\right]_{x=0}^{x=-1}$$

$$= \left(0 + \frac{1}{3}\right) - \left(\frac{1}{3} - 0\right)$$

$$\int_{CA} y^2\,dx - x^2\,dy = 0 \quad \text{——} (4)$$

Sub. (2), (3) and (4) in (1), we get

$$\int_{C} y^2\,dx - x^2\,dy = 0 - \frac{2}{3} + 0$$

$$= -\frac{2}{3}$$

→ If $\bar{F} = (x^2+y^2)i - 2xy\,j$ evaluate $\oint_C \bar{F}\cdot d\bar{r}$ where curve C is the rectangle in xy-plane bounded by $x=0$, $x=a$, $y=0$, $y=b$.

Sol: Given that $\bar{F} = (x^2+y^2)i - 2xy\,j$

We have to find $\oint_C \bar{F}\cdot d\bar{r}$

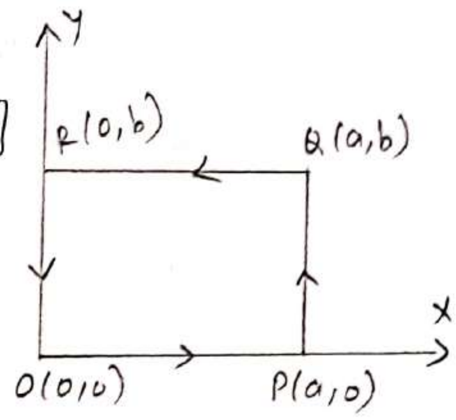
Let $\bar{r} = xi+yj+zk$

$\bar{r} = xi+yj$ [∵ In xy-plane $z=0$]

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = [(x^2+y^2)i - 2xy\,j]\,[dx\,i + dy\,j]$

$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx - 2xy\,dy$ ——— ①



The curve C is the rectangle bounded by $x=0$ $x=a$, $y=0$, $y=b$.

The vertices of the rectangle are $O(0,0)$ $P(a,0)$ $Q(a,b)$ $R(0,b)$.

$$\oint \bar{F}\cdot d\bar{r} = \int_{OP} \bar{F}\cdot d\bar{r} + \int_{PQ} \bar{F}\cdot d\bar{r} + \int_{QR} \bar{F}\cdot d\bar{r} + \int_{RO} \bar{F}\cdot d\bar{r} \quad \text{——— ②}$$

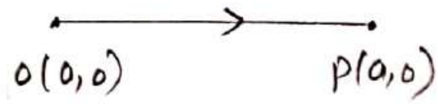
Case (i): To evaluate $\int_{OP} \bar{F}\cdot d\bar{r}$ (or) Along the line OP :-

We have $O(0,0)$ $P(a,0)$

Here $y=0 \Rightarrow dy=0$.

x varies from 0 to a.

∴ x limits are $x=0$, $x=a$.



From ①, $\bar{F}\cdot d\bar{r} = x^2\,dx$.

$$\int_{OP} \bar{F}\cdot d\bar{r} = \int_{OP} x^2\,dx$$

$$= \int_{x=0}^{x=a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{x=a}$$

$$= \frac{a^3}{3}.$$

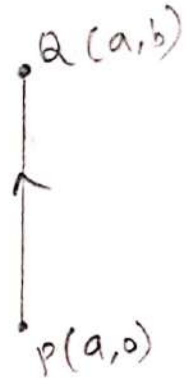
Case (ii):- To evaluate. $\int_{PQ} \bar{F}.d\bar{r}$ (or) Along the line PQ:

We have $P(a,0)$ $Q(a,b)$

Here $x=a \Rightarrow dx=0$.

y varies from 0 to b.

$\therefore$ y limits are $y=0$, $y=b$.



From (1), $\bar{F}.d\bar{r} = -2ay\,dy$

$$\int_{PQ} \bar{F}.d\bar{r} = \int_{PQ} -2ay\,dy$$

$$= \int_{y=0}^{y=b} -2ay\,dy$$

$$= \left[-2a.\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$= -ab^2 \quad \text{--- (4)}$$

Case (iii): To evaluate. $\int_{QR} \bar{F}.d\bar{r}$ (or) Along the line QR.

We have $Q(a,b)$ $R(0,b)$

Here $y=b \Rightarrow dy=0$

x varies from a to 0.

$\therefore$ x limits are $x=a$, $x=0$.

From (1). $\bar{F}.d\bar{r} = (x^2+b^2)\,dx$.

$$\int_{QR} \bar{F}.d\bar{r} = \int_{QR} (x^2+b^2)\,dx$$

$$= \int_{x=a}^{x=0} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2x\right]_{x=a}^{x=0}$$

$$= 0 - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -\frac{a^3}{3} - ab^2 \quad \text{--- (5)}$$

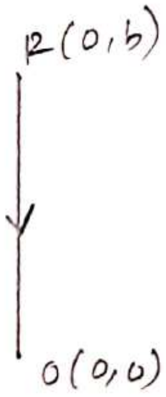
Case (iv) To evaluate $\int_{RO} \bar{F}\cdot d\bar{r}$ (or) Along the line RO :-

We have $R(0,b)$ $O(0,0)$

Here $x=0 \Rightarrow dx=0$.

y varies from b to 0.

$\therefore$ y limits $y=b, y=0$.



From (1), $\bar{F}\cdot d\bar{r}=0$

$$\int_{RO} \bar{F}\cdot d\bar{r} = 0 \quad \text{——— (6)}$$

Sub (3) (4). (5) and (6) in (2), we get

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$\oint_C \bar{F}\cdot d\bar{r} = -2ab^2$$

→ Evaluate the line integral $\int_C [(x^2+xy)dx + (x^2+y^2)dy]$ where C is the square formed by the lines $x=\pm 1$ and $y=\pm 1$ Ans: 0.

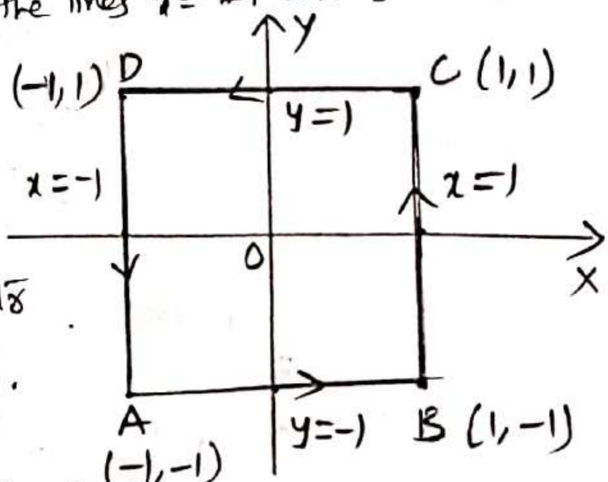
Sol: $I = \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

Let $\bar{F}\cdot d\bar{r} = (x^2+xy)\,dx + (x^2+y^2)\,dy$ ——— (1)

Given that C is the square formed by the lines $x=\pm 1$ and $y=\pm 1$.

The vertices of the square are

$A(-1,-1)$ $B(1,-1)$ $C(1,1)$ $D(-1,1)$



$$\int_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$$

——— (2).

Case (i) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(-1,-1)$ $B(1,-1)$

Here $y=-1 \Rightarrow dy=0$

x varies from -1 to 1

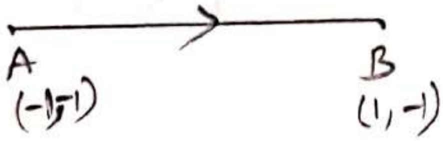
x limits $x=-1$, $x=1$

From (1), $\bar{F}\cdot d\bar{r} = (x^2-x)\,dx$ $[\because y=-1, dy=0]$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} (x^2 - x)\,dx$$

$$= \int_{x=-1}^{x=1} (x^2 - x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x=-1}^{x=1}$$



$$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{-1}{3} - \frac{1}{2}\right)$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \frac{2}{3} \quad \text{——— (3)}$$

Case (ii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :—

We have B (1,-1) C (1,1).

Here $x = 1 \Rightarrow dx = 0$.

y varies from -1 to 1.

$\therefore$ y limits $y = -1$, $y = 1$.

From (1) $\bar{F}\cdot d\bar{r} = (y^2+1)\,dy$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (y^2+1)\,dy$$

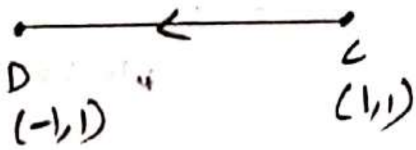
$$= \int_{y=-1}^{y=1} (y^2+1)\,dy$$

$$= \left[\frac{y^3}{3} + y\right]_{y=-1}^{y=1} = \left(\frac{1}{3}+1\right) - \left(\frac{-1}{3} - 1\right).$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \frac{8}{3} \quad \text{——— (4)}$$

Case (iii) To evaluate $\int_{CD} \bar{F}\cdot d\bar{r}$ (or) Along the line CD :—

We have C (1,1) D (-1,1)



Here $y = 1 \Rightarrow dy = 0$

x varies from -1 to 1.

$\therefore$ x limits $x = -1$, $x = 1$.

From (1). $\bar{F}\cdot d\bar{r} = (x^2 + x)\,dx$.

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} (x^2+x)\,dx$$

$$= \int_{x=-1}^{x=1} (x^2+x)\,dx$$

$$= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{x=-1}^{x=1} = \left(\frac{1}{3}+\frac{1}{2}\right)-\left(\frac{-1}{3}+\frac{1}{2}\right)$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = -\frac{2}{3} \quad \text{—— (5)}$$

<u>Case (iv)</u> <u>To evaluate $\int_{DA} \bar{F}\cdot d\bar{r}$ (or) Along the line DA</u> :-

We have $D(-1,1)$ $A(-1,-1)$

Here $x=-1 \Rightarrow dx=0$.

y varies from 1 to -1.

$\therefore$ y limits $y=1$, $y=-1$

From (1), $\bar{F}\cdot d\bar{r} = (y^2+1)\,dy$.

$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_{DA} (y^2+1)\,dy$$

$$= \int_{y=1}^{y=-1} (y^2+1)\,dy$$

$$= \left[\frac{y^3}{3}+y\right]_{y=1}^{y=-1}$$

$$= \left(\frac{-1}{3}-1\right)-\left(\frac{1}{3}+1\right)$$

$$\int_{DA} \bar{F}\cdot d\bar{r} = -\frac{8}{3} \quad \text{—— (6)}$$

Sub. (3) (4) (5) and (6) in (2), we get

$$\int_{C} \bar{F}\cdot d\bar{r} = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3}$$

$$\int_{C} \bar{F}\cdot d\bar{r} = 0.$$

Green's Theorem in a plane :-

(Transformation Between Line Integral and Double Integral)

If R is a closed region in xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Where C is traversed in the positive (anti clockwise) direction

Note:-

(i) Green's theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem.

(ii) Green's theorem in a vector notation.

Let $\bar{F} = M\hat{i} + N\hat{j}$ and $\bar{r} = x\hat{i} + y\hat{j}$

$\bar{F} \cdot d\bar{r} = M dx + N dy$

$$\text{curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \iint_R \text{curl}\,\bar{F} \cdot \bar{n}\, dR$$

Where $\bar{n} = \bar{k}$ for xy plane and $dR = dx\,dy$ and $\text{curl}\,\bar{F} \cdot \bar{n} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

(iii) Area of the plane region R bounded by a simple closed curve C.

Let $N = x$ and $M = -y$

Then $\oint_C x\,dy - y\,dx = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy$

Hence $A = \frac{1}{2}\oint_C x\,dy - y\,dx$.

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1) Verify Green's theorem in plane for $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the region bounded by $y=\sqrt{x}$ and $y=x^2$.

Sol:- Given that the integral $I = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ —— (1)

WKT Green's Theorem in a plane.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—— (2)}$$

Compare (1) with L.H.S of (2), we get

$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy.$$

Given that the region bounded by the curves $\underbrace{y=\sqrt{x}}_{(3)}$ and $\underbrace{y=x^2}_{(4)}$

solving (3) and (4), we get

$$y=\sqrt{x}, \quad y = x^2$$

$$x^2 = \sqrt{x} \Rightarrow x^4 = x$$

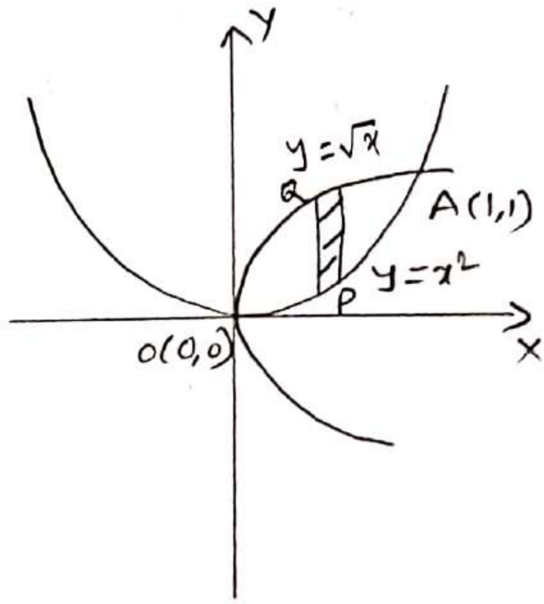
$$x(x^3-1) = 0$$

$$x = 0, 1$$

when $x=0$, $y=0$.

when $x=1$, $y=1$

The points of Intersection of the curves (3) and (4) is. $O(0,0)$ $A(1,1)$.



To evaluate. $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y.$$

Draw a vertical strip PQ in the region.

We have to fix x first.

In the region x varies from 0 to 1.

$\therefore$ x limits are $x=0$, $x=1$.

For each x, y varies a point P on parabola $y = x^2$ to a point Q on the parabola $y = \sqrt{x}$.

$\therefore$ y limits are $y = x^2$ and $y = \sqrt{x}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} 10y \, dy \, dx$$

$$= \int_{x=0}^{x=1} \left[10 \cdot \frac{y^2}{2}\right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= 5 \int_{x=0}^{x=1} \left[(\sqrt{x})^2 - (x^2)^2\right] dx$$

$$= 5 \int_{x=0}^{x=1} (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_{x=0}^{x=1}$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{3}{2} \quad \text{—— (5)}$$

<u>To evaluate $\oint_C M\,dx + N\,dy$:-</u>

The region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

To evaluate the line integral $\oint_C M\,dx + N\,dy$.

We can write $\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$ —— (6)

<u>Case (i)</u> :- To evaluate $\int_{OA} M\,dx + N\,dy$ (OR) Along the curve $y = x^2$.

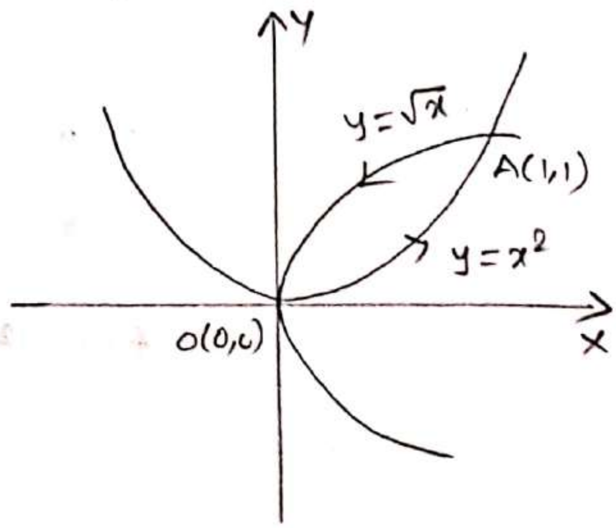
We have $y = x^2$

$dy = 2x\,dx$.

We have $O(0,0)$ $A(1,1)$

Here x varies from 0 to 1.

$\therefore$ x limits are $x = 0$, $x = 1$.



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$$\int_{OA} Mdx + Ndy = (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy \quad \left[\because y = x^2,\ dy = 2x\,dx\right]$$

$$= \left(3x^2 - 8(x^2)^2\right)dx + \left(4x^2 - 6x \cdot x^2\right)(2x\,dx)$$

$$Mdx + Ndy = \left(3x^2 + 8x^3 - 20x^4\right)dx .$$

$$\int_{OA} Mdx + Ndy = \int_{OA} \left(3x^2 + 8x^3 - 20x^4\right)dx$$

$$= \int_{x=0}^{x=1} \left(3x^2 + 8x^3 - 20x^4\right)dx$$

$$= \left[3 \cdot \frac{x^3}{3} + 8 \cdot \frac{x^4}{4} - 20 \cdot \frac{x^5}{5}\right]_{x=0}^{x=1}$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_{x=0}^{x=1}$$

$$= (1 + 2 - 4) - 0$$

$$\int_{OA} Mdx + Ndy = -1 \quad \text{——} \ (1)$$

case (ii) :- To evaluate $\int_{AO} Mdx + Ndy$ (or) Along the curve $y = \sqrt{x}$.

We have $y = \sqrt{x}$

i.e $x = y^2$

$dx = 2y\,dy$

We have ~~O(0,0)~~ $A(1,1)$ $O(0,0)$

Here y varies from 1 to 0

$\therefore$ y limits are $y = 1$, $y = 0$

$$Mdx + Ndy = (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= (3y^4 - 8y^2)(2y\,dy) + (4y - 6y \cdot y^2)\,dy \quad \left[\because x = y^2,\ dx = 2y\,dy\right]$$

$$= \left[6y^5 - 16y^3 + 4y - 6y^3\right]dy$$

$$Mdx + Ndy = (6y^5 + 4y - 22y^3)\,dy$$

$$Mdx + Ndy = (6y^5 + 4y - 22y^3)\,dy$$

$$\int_{AO} M\,dx + N\,dy = \int_{AO} (6y^5 + 4y - 22y^3)\,dy$$

$$= \int_{y=1}^{y=0} (6y^5 + 4y - 22y^3)\,dy$$

$$= \left[6 \cdot \frac{y^6}{6} + 4\frac{y^2}{2} - 22\frac{y^4}{4}\right]_{y=1}^{y=0}$$

$$= \left[y^6 + 2y^2 - \frac{11}{2}y^4\right]_{y=1}^{y=0}$$

$$= 0 - \left(1 + 2 - \frac{11}{2}\right)$$

$$\int_{AO} M\,dx + N\,dy = \frac{5}{2} \quad \text{--- (8)}$$

sub. (7) and (8) in (6), we get

$$\oint_C M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2} \quad \text{--- (9)}$$

$\therefore$ From (5) and (9),

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

2) Verify Green's theorem for $\oint_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$ where C is the closed curve in xy-plane bounded by the curves $y = x^2$ and $y^2 = x$. Ans:- $\frac{1}{30}$.

Sol:- Given that $I = \oint_C (2xy - x^2)\,dx + (x + y^2)\,dy$ --- (1)

WKT Green's Theorem $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ --- (2)

compare (1) with L.H.S of (2), we get

$$M = 2xy - x^2 \qquad N = x + y^2$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 1.$$

The region bounded by the curves $y = x^2$ and $y^2 = x$.

Solving $y = x^2$, $y^2 = x$

$$y^2 = x \implies x^4 = x$$

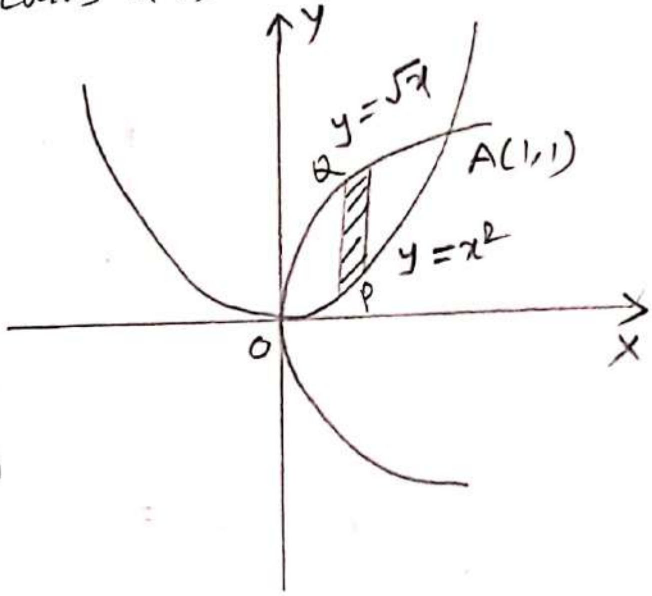
$$x(x^3 - 1) = 0$$

$$x = 0, 1$$

When $x=0,\ y=0$.

When $x=1,\ y=1$.

$\therefore$ The points of intersection of the curves $O(0,0)$ $A(1,1)$



To evaluate $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy$

$M = 2xy - x^2 \qquad N = x + y^2$

$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - 2x.$

Draw a vertical strip PQ in the region

We have to fix x first.

In the region x varies from 0 to 1.

$\therefore$ x limits $x=0,\ x=1$.

For each x, y varies from a point P on $y=x^2$ to a point Q on $y^2=x$.

$\therefore$ y limits $y=x^2,\ y=\sqrt{x}$.

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}} (1-2x)\,dy\,dx.$$

$$= \int_{x=0}^{x=1} (1-2x)\,[y]_{y=x^2}^{y=\sqrt{x}}\,dx.$$

$$= \int_{x=0}^{x=1} (1-2x)(\sqrt{x}-x^2)\,dx$$

$$= \int_{x=0}^{x=1} \left(\sqrt{x} - 2x^{3/2} - x^2 + 2x^3\right) dx.$$

$$= \left[\frac{2}{3}x^{3/2} - 2\cdot\frac{2}{5}x^{5/2} - \frac{x^3}{3} + 2\cdot\frac{x^4}{4}\right]_{x=0}^{x=1}$$

$$= \frac{1}{2} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2}$$

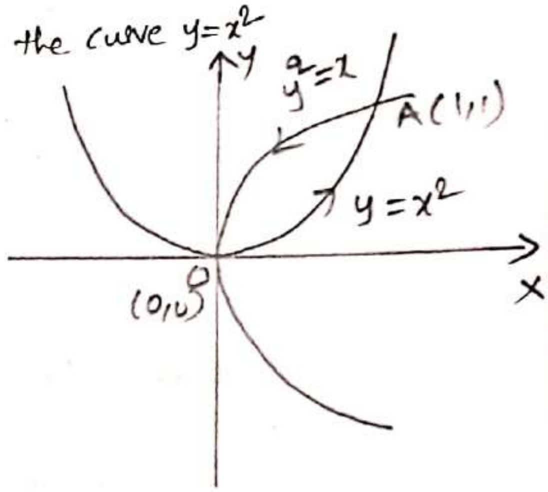
$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy = \frac{1}{30}$$

To evaluate. $\oint_C M\,dx + N\,dy$:-

To evaluate the line integral $\oint_C M\,dx + N\,dy$

We can write. $\oint_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AO} M\,dx+N\,dy$ —— (3)

Case (i) :- To evaluate $\int_{OA} M\,dx+N\,dy$ (OR) Along the curve $y=x^2$



We have $y=x^2$

$dy = 2x\,dx$

We have $O(0,0)$ $A(1,1)$

x varies from 0 to 1.

$\therefore$ x limits $x=0, x=1$.

$M\,dx+N\,dy = (2xy-x^2)\,dx + (x+y^2)\,dy$

$= (2x^3-x^2)\,dx + (x^2+x^4)\,2x\,dx$

$M\,dx+N\,dy = (2x^3+x^2+2x^5)\,dx$

$\int_{OA} M\,dx+N\,dy = \int_{OA} (2x^3+x^2+2x^5)\,dx$

$= \int_{x=0}^{x=1} (2x^3+x^2+2x^5)\,dx$

$= \left[\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\right]_{x=0}^{x=1}$

$\int_{OA} M\,dx+N\,dy = \frac{7}{6}$ —— (4)

Case (ii) To evaluate $\int_{AO} M\,dx+N\,dy$ (OR) Along the curve $y^2=x$.

We have $x=y^2$

$dx = 2y\,dy$

We have $A(1,1)$ $O(0,0)$

y varies from 1 to 0.

$\therefore$ y limits $y=1, y=0$.

$M\,dx+N\,dy = (2xy-x^2)\,dx + (x+y^2)\,dy$

$= (2y^3-y^4)\,2y\,dy + (y^2+y^2)\,dy$

$$M\,dx + N\,dy = (4y^4 - 2y^5 + 2y^2)\,dy.$$

$$\int_{AO} M\,dx + N\,dy = \int_{AO} (4y^4 - 2y^5 + 2y^2)\,dy$$

$$= \int_{y=1}^{y=0} (4y^4 - 2y^5 + 2y^2)\,dy$$

$$= \left[4 \cdot \frac{y^5}{5} - 2\frac{y^6}{6} + 2\frac{y^3}{3}\right]_{y=1}^{y=0}$$

$$= \frac{4}{5} + \frac{1}{3} - \frac{2}{3}$$

$$\int_{AO} M\,dx + N\,dy = \frac{-17}{15} \quad \text{— (5)}$$

sub (4) and (5) in (3), we get

$$\oint_C M\,dx + N\,dy = \frac{7}{6} - \frac{17}{15} = \frac{1}{30}$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

3) Verify Green's theorem for $\int_C [(xy+y^2)\,dx + x^2\,dy]$ where c is bounded by $y=x$ and $y=x^2$

Ans: $-\frac{1}{20}$.

sol:- Given that $I = \int_C [(xy+y^2)\,dx + x^2\,dy]$ —— ①

Wkt Greens theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \text{ —— ②}$$

Compare ① with L.H.S ② we get

$$M = xy + y^2 \qquad N = x^2.$$

The region is bounded by $y=x$ —— ① and $y=x^2$ —— ②

solving ① and ②,

$$y = x^2 \quad \text{and} \quad y = x$$

$$x^2 = x$$

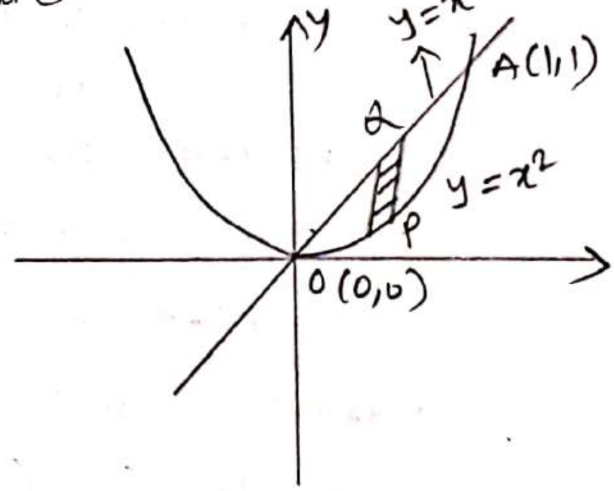
$$x(x-1) = 0$$

$$x=0,\ x=1.$$

When $x=0$, $y=0$

When $x=1$, $y=1$.

∴ The points of intersection of ① and ② is $O(0,0)$ and $A(1,1)$



<u>To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$:-</u>

$$M = xy + y^2 \qquad N = x^2$$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - 2y$$

Draw a vertical strip PQ in the region

We have to fix x first

In the region x varies from 0 to 1.

∴ x limits $x=0$, $x=1$.

For each x, y varies from a point P on the parabola $y=x^2$ to a point Q on the line $y=x$. ∴ y limits $y=x^2$, $y=x$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{x=1} \left[xy - 2.\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{x=1} (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{x=0}^{x=1}$$

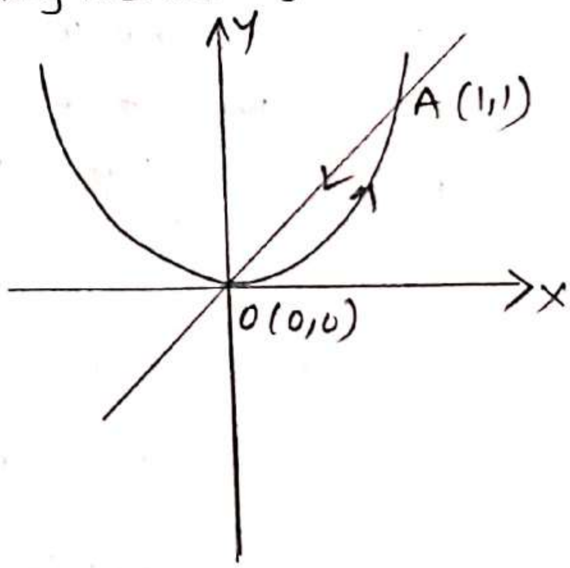
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{-1}{20} \quad \text{--- (3)}$$

<u>To evaluate</u> $\oint_C$ <u>$Mdx + Ndy$</u> :-

To evaluate the line integral $\oint_C Mdx + Ndy$

We can write $\oint_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AO} Mdx + Ndy$ --- (4)

<u>Case (i)</u> To evaluate $\int_{OA} Mdx + Ndy$ (or) Along the curve $y = x^2$



We have $y = x^2$

$dy = 2x\,dx$.

We have $O(0,0)$ $A(1,1)$

x varies from 0 to 1

$\therefore$ x limits $x=0, x=1$.

$Mdx + Ndy = (xy + y^2)\,dx + x^2\,dy$

$Mdx + Ndy = (x^3 + x^4 + 2x^3)\,dx$

$= (3x^3 + x^4)\,dx$

$$\int_{OA} Mdx + Ndy = \int_{OA} (3x^3 + x^4)\,dx$$

$$= \int_{x=0}^{x=1} (3x^3 + x^4)\,dx$$

$$\int_{OA} M dx + N dy = \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_{x=0}^{x=1}$$

$$= \frac{3}{4} + \frac{1}{5}$$

$$= \frac{19}{20} \quad \text{--- (5)}$$

Case(ii) To evaluate $\int_{AO} M dx + N dy$ (or) Along the line $y = x$.

We have. $y = x \implies dy = dx$

We have $A(1,1)$ $O(0,0)$

$\because x$ varies from 1 to 0

$\therefore$ x limits $x = 1, x = 0$

$$M dx + N dy = (xy + y^2) dx + x^2 dy$$

$$= 3x^2 dx$$

$$\int_{AO} M dx + N dy = \int_{AO} 3x^2 dx$$

$$= \int_{x=1}^{x=0} 3x^2 dx$$

$$= \left[3 \cdot \frac{x^3}{3}\right]_{x=1}^{x=0}$$

$$\int_{AO} M dx + N dy = 3\left(-\frac{1}{3}\right) = -1 \quad \text{--- (6)}$$

sub (5) & (6) in (4), we get

$$\oint_C M dx + N dy = \frac{19}{20} - 1 = \frac{-1}{20}$$

$$\therefore \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

→ Verify Green's theorem for $\oint_C (y-\sin x)\,dx + \cos x\,dy$ where c is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $\pi y = 2x$. (OR)

Verify Green's theorem for $\oint_C (y-\sin x)\,dx + \cos x\,dy$ where c is the boundary of the triangle in xy-plane whose vertices are $(0,0)$ $(\frac{\pi}{2},0)$ and $(\frac{\pi}{2},1)$ traversed in the positive direction.

Sol:- Given that $I = \oint_C (y-\sin x)\,dx + \cos x\,dy$ —— ①

W.K.T Green's theorem in a plane.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \text{ —— ②}$$

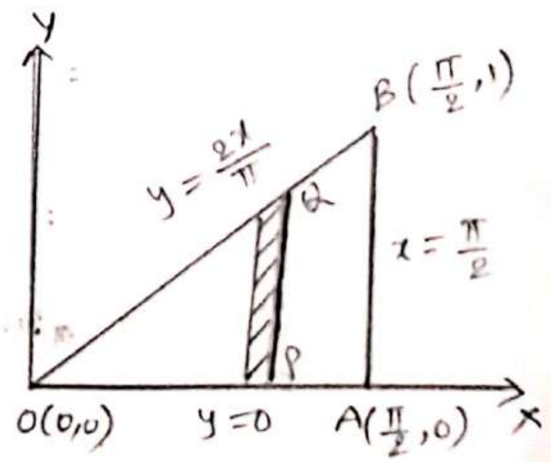
Compare ① with L.H.S of ②, we get

Here $M = y - \sin x$ $N = \cos x$.

Given that c is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $y=\frac{2x}{\pi}$

The vertices of the triangle are $O(0,0)$ $A(\frac{\pi}{2},0)$ $B(\frac{\pi}{2},1)$.

To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$:-



$M = y - \sin x$ $N = \cos x$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = -\sin x$.

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\sin x - 1$.

Draw a vertical strip PQ in the region

We have to fix x first

In the region, x varies from 0 to $\frac{\pi}{2}$.

$\therefore$ x limits are $x=0$, $x=\frac{\pi}{2}$

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $y=\frac{2x}{\pi}$.

$\therefore$ y limits are $y=0$, $y=\frac{2x}{\pi}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-1-\sin x)\,dx\,dy$$

$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{y=\frac{2x}{\pi}} (-1-\sin x)\,dy\,dx.$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} (-1-\sin x)\,[y]_{y=0}^{y=\frac{2x}{\pi}}\,dx$$

$$= -\int_{x=0}^{x=\pi/2} (1+\sin x)\,\frac{2x}{\pi}\,dx$$

$$= -\frac{2}{\pi}\int_{x=0}^{x=\pi/2} (x + x\sin x)\,dx$$

$$= -\frac{2}{\pi}\left[\int_{x=0}^{x=\pi/2} x\,dx + \int_{x=0}^{x=\pi/2} x\sin x\,dx\right]$$

$$= -\frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_{x=0}^{x=\pi/2} + \left[x(-\cos x) - 1.(-\sin x)\right]_{x=0}^{x=\frac{\pi}{2}}\right]$$

$$= -\frac{2}{\pi}\left[\left[\frac{\pi^2}{8} - 0\right] + \left\{\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - 0\right\}\right]$$

$$= -\frac{2}{\pi}\left(\frac{\pi^2}{8} + 1\right)$$

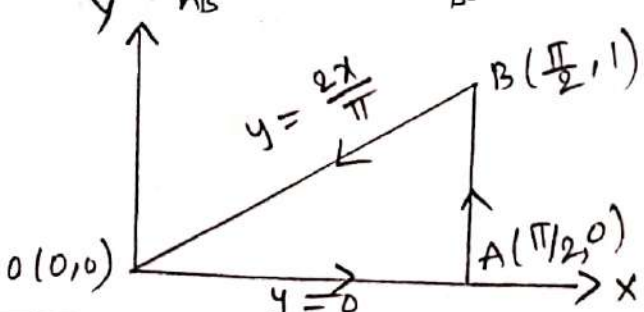
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right) \quad \text{--- (3)}$$

To evaluate $\oint_C M\,dx + N\,dy$:-

The region is ΔOAB.

To evaluate the line integral $\oint_C M\,dx + N\,dy$. --- (4)

We can write $\oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AB} M\,dx + N\,dy + \int_{BO} M\,dx + N\,dy$



Case (i) :- To evaluate $\int_{OA} M\,dx + N\,dy$ (OR) Along the line OA :-

We have $O(0,0)$ $A(\frac{\pi}{2},0)$

Here $y=0 \Rightarrow dy=0$.

x varies from 0 to $\frac{\pi}{2}$.

$\therefore$ x limits are $x=0$, $x=\frac{\pi}{2}$

$$M\,dx + N\,dy = (y-\sin x)\,dx + \cos x\,dy$$

$$M\,dx + N\,dy = -\sin x\,dx \quad [\because y=0,\ dy=0]$$

$$\int_{OA} M\,dx + N\,dy = \int_{OA} -\sin x\,dx$$

$$= \int_{x=0}^{x=\pi/2} -\sin x\,dx$$

$$= [\cos x]_{x=0}^{x=\pi/2} = \cos\frac{\pi}{2} - \cos 0.$$

$$\int_{OA} M\,dx + N\,dy = -1 \quad \text{—— (5)}.$$

Case (ii) :- To evaluate $\int_{AB} M\,dx + N\,dy$ (OR) Along the line AB :-

We have $A(\frac{\pi}{2},0)$ $B(\frac{\pi}{2},1)$

Here $x=\frac{\pi}{2} \Rightarrow dx=0$.

y varies from 0 to 1.

$\therefore$ y limits are $y=0, y=1$.

$$M\,dx + N\,dy = (y-\sin x)\,dx + \cos x\,dy$$

$$M\,dx + N\,dy = (y-\sin\frac{\pi}{2})(0) + \cos\frac{\pi}{2}\cdot dy = 0 \quad [\because x=\frac{\pi}{2},\ dx=0]$$

$$\int_{AB} M\,dx + N\,dy = \int_{AB} 0 = 0 \quad \text{—— (6)}.$$

Case (iii) :- To evaluate $\int_{BO} M\,dx + N\,dy$ (OR) Along the line BO :

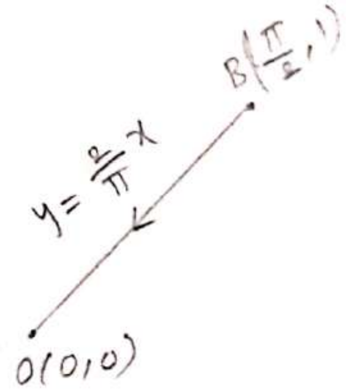
An equation of the line BO is $y = \frac{2x}{\pi}$

$$dy = \frac{2}{\pi}\,dx.$$

We have $B(\frac{\pi}{2},1)$ $O(0,0)$

x varies from $\frac{\pi}{2}$ to 0.

$\therefore$ x limits are $x=\frac{\pi}{2}, x=0$.



$$Mdx+Ndy = (y-\sin x)dx + \cos x\,dy$$

$$= \left(\frac{2x}{\pi}-\sin x\right)dx + \frac{2}{\pi}\cos x\,dx$$

$$= \left[\frac{2x}{\pi} - \sin x + \frac{2}{\pi}\cos x\right]dx$$

$$\int_{BO} Mdx+Ndy = \int_{BO}\left[\frac{2x}{\pi} - \sin x + \frac{2}{\pi}\cos x\right]dx$$

$$= \int_{x=\frac{\pi}{2}}^{x=0}\left[\frac{2x}{\pi} - \sin x + \frac{2}{\pi}\cos x\right]dx$$

$$= \left[\frac{2}{\pi}\cdot\frac{x^2}{2} + \cos x + \frac{2}{\pi}\sin x\right]_{x=\frac{\pi}{2}}^{x=0}$$

$$= 1 - \left\{\frac{2}{\pi}\cdot\frac{1}{2}\frac{\pi^2}{4} + \cos\frac{\pi}{2} + \frac{2}{\pi}\sin\frac{\pi}{2}\right\}$$

$$\int_{BO} Mdx+Ndy = 1-\frac{\pi}{4}-\frac{2}{\pi} \quad \text{—— (7)}$$

sub. (5) (6) and (7) in (4), we get

$$\oint_C Mdx+Ndy = -1+0+1-\frac{\pi}{4}-\frac{2}{\pi}$$

$$\oint_C Mdx+Ndy = -\left(\frac{\pi}{4}+\frac{2}{\pi}\right) \quad \text{—— (8)}$$

$\therefore$ From (4) and (8)

$$\therefore \oint_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$\therefore$ Green's theorem verified.

→ Verify Green's theorem for $\int_c (e^{-x}\sin y)\,dx + (e^{-x}\cos y)\,dy$ Where c is the boundary of the rectangle whose vertices are $(0,0)$ $(\pi,0)$ $(\pi,\frac{\pi}{2})$ and $(0,\frac{\pi}{2})$ traversed in the positive direction.

Sol:- Given that $I = \int_c (e^{-x}\sin y)\,dx + (e^{-x}\cos y)\,dy$ — ①

WKT Green's theorem in a plane.

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{— ②}$$

Compare ① with L.H.S of ②, we get

$$M = e^{-x}\sin y \qquad N = e^{-x}\cos y.$$

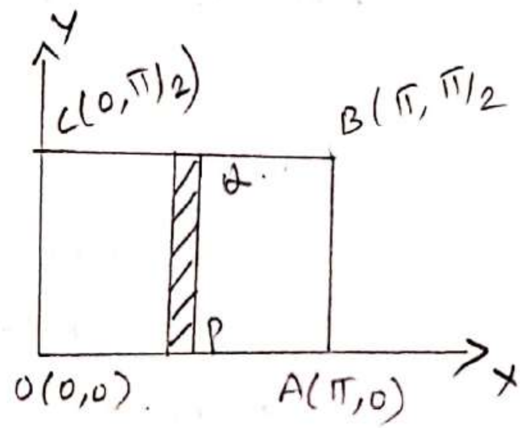
Given that c is the boundary of the rectangle whose vertices are $O(0,0)$ $A(\pi,0)$ $B(\pi,\frac{\pi}{2})$ and $C(0,\frac{\pi}{2})$.

To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$M = e^{-x}\sin y \qquad N = e^{-x}\cos y$$

$$\frac{\partial M}{\partial y} = e^{-x}\cos y \qquad \frac{\partial N}{\partial x} = -e^{-x}\cos y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y = -2e^{-x}\cos y$$



Draw a vertical strip PQ in the region.

We have to fix x first.

In the region x varies from 0 to π ∴ x limits are $x=0$, $x=\pi$

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $y=\frac{\pi}{2}$.

∴ y limits are $y=0$, $y=\frac{\pi}{2}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R -2e^{-x}\cos y\,dx\,dy$$

$$= \int_{y=0}^{y=\pi/2}\int_{x=0}^{x=\pi} -2e^{-x}\cos y\,dx.dy$$

$$= -2\int_{y=0}^{y=\pi/2}\left[\frac{e^{-x}}{-1}\right]_{x=0}^{x=\pi} \cos y\,dy$$

$$= 2\int_{y=0}^{y=\pi/2} (e^{-\pi}-1)\cos y\, dy$$

$$= 2(e^{-\pi}-1)\left[\sin y\right]_{y=0}^{y=\pi/2}$$

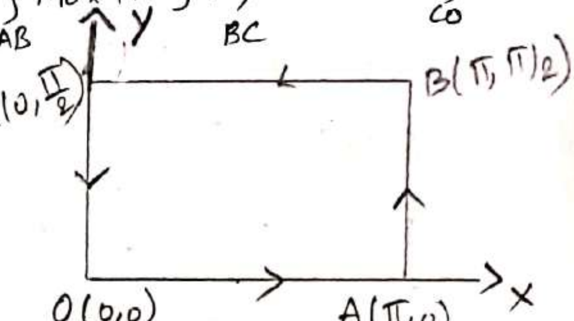
$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = 2(e^{-\pi}-1) \quad \text{—— (3)}$$

<u>To evaluate $\oint_C M\,dx+N\,dy$</u> :-

The region is rectangle OABC.

To evaluate the line integral $\oint M\,dx+N\,dy$. — (4).

We can write $\oint_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AB} M\,dx+N\,dy + \int_{BC} M\,dx+N\,dy + \int_{CO} M\,dx+N\,dy$



<u>Case (i)</u>:- To evaluate $\int_{OA} M\,dx+N\,dy$ (OR)

Along the line OA :-

We have $O(0,0)$ $A(\pi,0)$.

Here $y=0 \Rightarrow dy=0$.

x varies from 0 to π.

$\therefore$ x limits are $x=0$, $x=\pi$

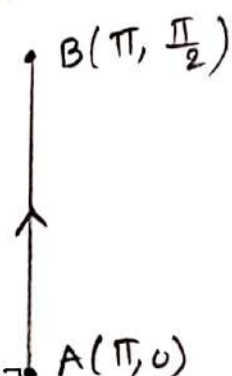
$M\,dx+N\,dy = e^{-x}\sin y\, dx + e^{-x}\cos y\, dy$. $[\because y=0,\ dy=0]$

$M\,dx+N\,dy = e^{-x}\sin(0)\,dx + e^{-x}\cos(0)\cdot 0$

$M\,dx+N\,dy = 0$.

$$\int_{OA} M\,dx+N\,dy = \int_{OA} 0. = 0 \quad \text{—— (5)}$$

<u>Case (ii)</u>: To evaluate $\int_{AB} M\,dx+N\,dy$ (OR) Along the line AB :-



We have $A(\pi,0)$ $B(\pi,\frac{\pi}{2})$.

Here $x=\pi \Rightarrow dx=0$.

y varies from 0 to $\frac{\pi}{2}$.

$\therefore$ y limits are $y=0$, $y=\frac{\pi}{2}$.

$M\,dx+N\,dy = e^{-x}\sin y\, dx + e^{-x}\cos y\, dy$ $[\because x=\pi,\ dx=0]$

$M\,dx+N\,dy = e^{-\pi}\sin y\,(0) + e^{-\pi}\cos y\,.dy = +e^{-\pi}\cos y\, dy$

$$\int_{AB} M\,dx + N\,dy = \int_{AB} +e^{-\pi}\cos y\,dy$$

$$= \int_{y=0}^{y=\frac{\pi}{2}} +e^{-\pi}\cos y\,dy$$

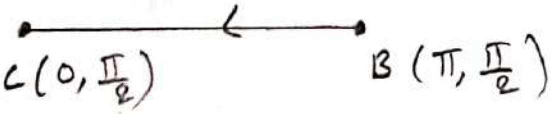
$$= \left[+e^{-\pi}\sin y\right]_{y=0}^{y=\frac{\pi}{2}}$$

$$= +e^{-\pi}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$\int_{AB} M\,dx + N\,dy = +e^{-\pi} \quad \text{——— (6)}$$

Case (iii) To evaluate $\int_{BC} M\,dx + N\,dy$ (OR) Along the line BC :-

We have $B(\pi, \frac{\pi}{2})$ $C(0, \frac{\pi}{2})$



Here $y = \frac{\pi}{2} \Rightarrow dy = 0$.

x varies from π to 0.

$\therefore$ x limits $x = \pi$, $x = 0$.

$$M\,dx + N\,dy = e^{-x}\sin y\,dx + e^{-x}\cos y\,dy \quad \left[\because y = \frac{\pi}{2},\ dy = 0\right]$$

$$= e^{-x}\sin\left(\frac{\pi}{2}\right)dx + e^{-x}\cos\left(\frac{\pi}{2}\right)\cdot 0$$

$$M\,dx + N\,dy = e^{-x}\,dx$$

$$\int_{BC} M\,dx + N\,dy = \int_{BC} e^{-x}\,dx$$

$$= \int_{x=\pi}^{x=0} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_{x=\pi}^{x=0}$$

$$= -\left[e^{0} - e^{-\pi}\right]$$

$$= e^{-\pi} - 1 \quad \text{——— (7)}$$

Case (iv) To evaluate $\int_{CO} M\,dx + N\,dy$ (OR) Along the line CO :-

We have $C(0, \frac{\pi}{2})$ $O(0,0)$

Here $x = 0 \Rightarrow dx = 0$

y varies from $\frac{\pi}{2}$ to 0.

$\therefore$ y limits $y = \frac{\pi}{2}$, $y = 0$.

$$M\,dx + N\,dy = e^{-x}\sin y\,dx + e^{-x}\cos y\,dy \quad [\because x=0,\ dx=0]$$

$$= e^{-0}\sin y\,(0) + e^{-0}\cos y\,dy$$

$$M\,dx + N\,dy = \cos y\,dy$$

$$\int_{CO} M\,dx + N\,dy = \int_{CO} \cos y . dy$$

$$= \int_{y=\frac{\pi}{2}}^{y=0} \cos y\,dy$$

$$= \left[\sin y\right]_{y=\frac{\pi}{2}}^{y=0}$$

$$\int_{CO} M\,dx + N\,dy = -1 \quad \text{——} \ (8)$$

Sub. (5), (6) (7) and (8) in (4), we get

$$\oint_C M\,dx + N\,dy = e^{-\pi} + e^{-\pi} - 1 - 1$$

$$\oint_C M\,dx + N\,dy = 2(e^{-\pi} - 1) \quad \text{——} \ (9).$$

From (3) and (9)

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

→ Verify Green's theorem in the plane for $\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$

Where c is a square with vertices $(0,0)$ $(2,0)$ $(0,2)$ $(2,2)$

sol:- Given that $\int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy$ —— ①

WKT Green's theorem in a plane,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \text{ —— ②}$$

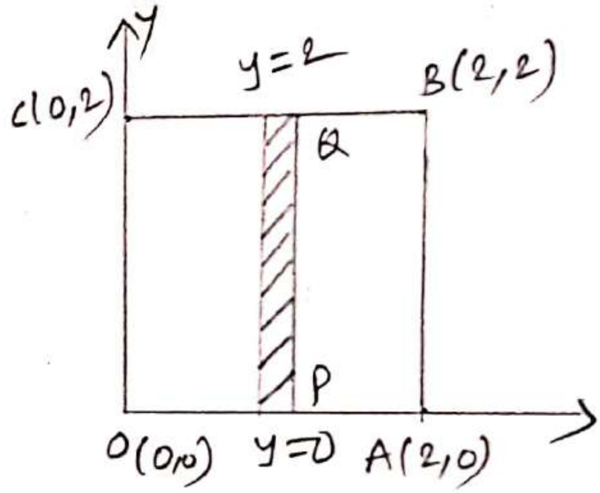
Compare ① with L.H.S of ②, we get

$$M = x^2 - xy^3 \qquad N = y^2 - 2xy$$

Given that c is the boundary of the square whose vertices are

$O(0,0)$ $A(2,0)$ $B(2,2)$ $C(0,2)$.

To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



$$M = x^2 - xy^3 \qquad N = y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3xy^2 - 2y.$$

Draw a vertical strip PQ in the region.

We have to fix x first

In the region x varies from 0 to 2

$\therefore$ x limits are $x=0, x=2$.

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $y=2$.

$\therefore$ y limits are $y=0, y=2$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (3xy^2 - 2y)\,dx\,dy$$

$$= \int_{y=0}^{y=2} \int_{x=0}^{x=2} (3xy^2 - 2y)\,dx\,dy$$

$$= \int_{y=0}^{y=2} \left[\frac{3x^2}{2}y^2 - 2xy\right]_{x=0}^{x=2} dy$$

$$= \int_{y=0}^{y=2} (6y^2 - 4y)\,dy$$

$$= \left[6\frac{y^3}{3} - 4\frac{y^2}{2}\right]_{y=0}^{y=2}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 8. \quad \text{——— (3)}$$

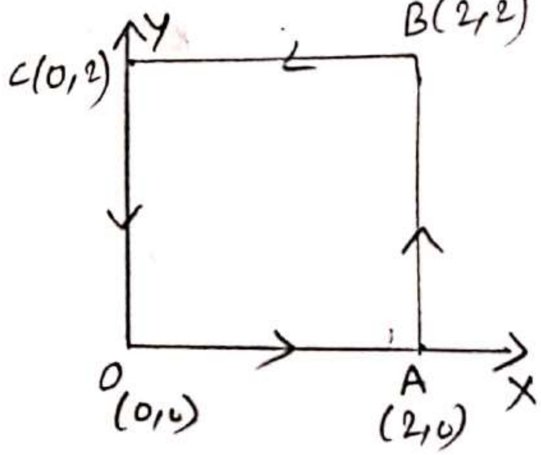
<u>To evaluate</u> $\oint_C M dx + N dy$:-

The region is square OABC.

To evaluate the line integral $\oint_C M dx + N dy$. ——— (4)

We can write $\oint_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AB} M dx + N dy + \int_{BC} M dx + N dy + \int_{CO} M dx + N dy$

<u>Case(i)</u> To evaluate $\int_{OA} M dx + N dy$ (OR) Along the line OA :-



We have O(0,0) A(2,0)

Here $y=0 \Rightarrow dy=0$

x varies from 0 to 2

$\therefore$ x limits $x=0,\ x=2$.

$$M dx + N dy = (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$M dx + N dy = x^2\,dx \qquad [\because y=0,\ dy=0]$$

$$\int_{OA} M dx + N dy = \int_{OA} x^2\,dx$$

$$= \int_{x=0}^{x=2} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{x=2}$$

$$\int_{OA} M dx + N dy = \frac{8}{3} \quad \text{——— (5)}$$

Case (ii) To evaluate $\int_{AB} M\,dx + N\,dy$ (OR) Along the line AB :-

We have $A(2,0)$ $B(2,2)$.

Here $x = 2 \Rightarrow dx = 0$

y varies from 0 to 2

$\therefore$ y limits $y=0$, $y=2$

$$M\,dx + N\,dy = (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$M\,dx + N\,dy = (y^2 - 4y)\,dy \quad [\because x = 2, dx = 0]$$

$$\int_{AB} M\,dx + N\,dy = \int_{AB} (y^2 - 4y)\,dy$$

$$= \int_{y=0}^{y=2} (y^2 - 4y)\,dy$$

$$= \left[\frac{y^3}{3} - 4\frac{y^2}{2}\right]_{y=0}^{y=2}$$

$$= \frac{8}{3} - 8$$

$$\int_{AB} M\,dx + N\,dy = -\frac{16}{3} \quad \text{--- (6)}$$

Case (iii) To evaluate $\int_{BC} M\,dx + N\,dy$ (OR) Along the line BC :-

We have $B(2,2)$ $C(0,2)$

Here $y = 2 \Rightarrow dy = 0$

x varies from 2 to 0

$\therefore$ x limits $x = 2$, $x = 0$.

$$M\,dx + N\,dy = (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$M\,dx + N\,dy = (x^2 - 8x)\,dx \quad [\because y = 2, dy = 0]$$

$$\int_{BC} M\,dx + N\,dy = \int_{BC} (x^2 - 8x)\,dx$$

$$= \int_{x=2}^{x=0} (x^2 - 8x)\,dx$$

$$= \left[\frac{x^3}{3} - 8\frac{x^2}{2}\right]_{x=2}^{x=0}$$

$$\int_{BC} M\,dx + N\,dy = +\frac{40}{3} \quad \text{--- (7)}$$

Case (iv) To evaluate $\int_{CO} Mdx + Ndy$ (OR) Along the line CO :-

We have $C(0,2)$ $O(0,0)$

Here $x=0 \Rightarrow dx=0$

y varies from 2 to 0.

$\therefore$ y limits $y=2, y=0$.

$$Mdx + Ndy = (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$Mdx + Ndy = y^2 dy$$

$$\int_{CO} Mdx + Ndy = \int_{CO} y^2 dy$$

$$= \int_{y=2}^{y=0} y^2 dy$$

$$= \left[\frac{y^3}{3}\right]_{y=2}^{y=0}$$

$$\int_{CO} Mdx + Ndy = -\frac{8}{3} \quad \text{—— (8)}$$

Sub. (5) (6) (7) and (8) in (4), we get

$$\oint_C Mdx + Ndy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3}$$

$$\oint_C Mdx + Ndy = 8$$

$$\therefore \oint_C Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Green's theorem verified.

Verify Green's theorem for $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ where C is the rectangle with vertices (0,0) (π,0) (π,1) (0,1). Ans: $\pi(\cosh 1 - 1)$

Sol:- Given that $\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$ —— ①

WKT Green's theorem in a plane.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—— ②}$$

Compare ① with L.H.S of ②, we get

$M = x^2 - \cosh y \qquad N = y + \sin x$.

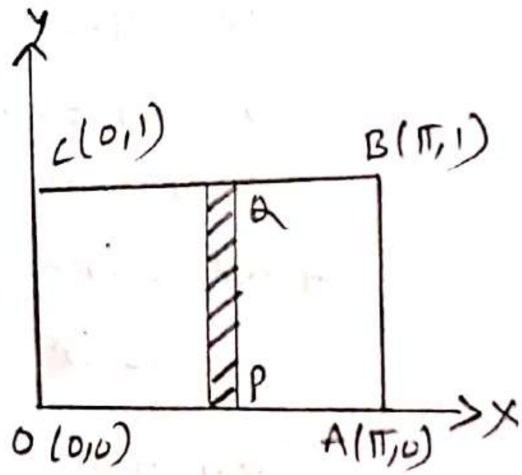
Given that C is the rectangle with vertices O(0,0) A(π,0) B(π,1) and C(0,1)

<u>To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$</u>

$M = x^2 - \cosh y \qquad N = y + \sin x$

$\frac{\partial M}{\partial y} = -\sinh y \qquad \frac{\partial N}{\partial x} = \cos x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \cos x + \sinh y$



Draw a vertical strip PQ in the region

We have to fix x first.

In the region x varies from 0 to π.

$\therefore$ x limits are $x = 0,\ x = \pi$.

For each x, y varies from a point P on x-axis (y=0) to a point Q on the line $y = 1$.

$\therefore$ y limits are $y = 0,\ y = 1$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (\cos x + \sinh y)\,dx\,dy$$

$$= \int_{x=0}^{x=\pi} \int_{y=0}^{y=1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_{x=0}^{x=\pi} \Big[\cos x \cdot y + \cosh y\Big]_{y=0}^{y=1} dx.$$

$$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \int_{x=0}^{x=\pi}(\cos x+\cosh 1-1)\,dx$$

$$= \Big[\sin x + x\cosh 1 - x\Big]_{x=0}^{x=\pi}$$

$$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \pi(\cosh 1-1) \quad \text{--- (3)}$$

<u>To evaluate $\oint_C M\,dx+N\,dy$</u> :-

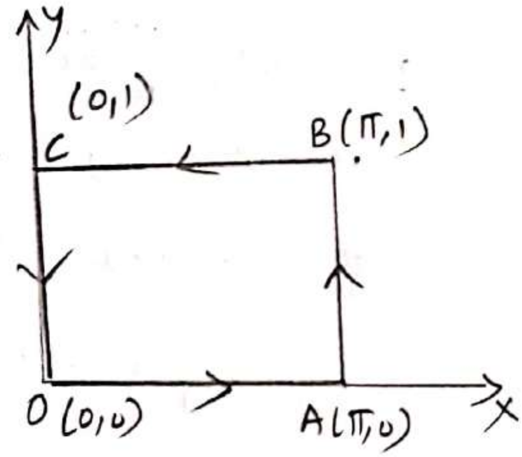
The region is rectangle OABC

To evaluate the line integral $\oint_C M\,dx+N\,dy$ — (4)

We can write $\oint_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AB} M\,dx+N\,dy + \int_{BC} M\,dx+N\,dy + \int_{CO} M\,dx+N\,dy$

<u>Case (i) To evaluate $\int_{OA} M\,dx+N\,dy$ (OR) Along the line OA</u> :-

We have $O(0,0)$ $A(\pi,0)$



Here $y=0 \Rightarrow dy=0$

x varies from 0 to π.

$\therefore$ x limits $x=0,\ x=\pi$.

$$M\,dx+N\,dy = (x^2-\cosh y)\,dx + (y+\sin x)\,dy$$

$$= (x^2-\cosh 0)\,dx \quad [\because y=0,\ dy=0]$$

$$\int_{OA} M\,dx+N\,dy = \int_{OA}(x^2-\cosh 0)\,dx$$

$$= \int_{x=0}^{x=\pi}(x^2-\cosh 0)\,dx$$

$$= \left[\frac{x^3}{3} - x\cosh 0\right]_{x=0}^{x=\pi}$$

$$\int_{OA} M\,dx+N\,dy = \frac{\pi^3}{3} - \pi \quad \text{--- (5)} \quad [\because \cosh 0 = 1]$$

<u>Case (ii) To evaluate $\int_{AB} M\,dx+N\,dy$ (OR) Along the line AB</u> :-

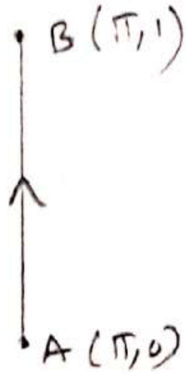
We have $A(\pi,0)$ $B(\pi,1)$

Here $x=\pi \Rightarrow dx=0$

y varies from 0 to 1.

$\therefore$ y limits $y=0, y=1$.

$$M\,dx + N\,dy = y\,dy \quad [\because x=\pi, dx=0]$$



$$\int_{AB} M\,dx + N\,dy = \int_{AB} y\,dy$$

$$= \int_{y=0}^{y=1} y\,dy$$

$$= \left[\frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$\int_{AB} M\,dx + N\,dy = \frac{1}{2} \quad \text{—— (6)}$$

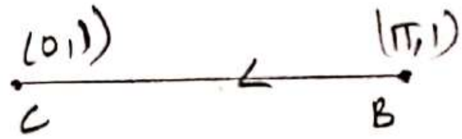
<u>Case (iii)</u> <u>To evaluate</u> $\underline{\int_{BC} M\,dx + N\,dy}$ <u>(OR) Along the line BC</u> :-

We have $B(\pi,1)$ $C(0,1)$

Here $y=1 \Rightarrow dy=0$

x varies from π to 0

$\therefore$ x limits $x=\pi, x=0$



$$M\,dx + N\,dy = (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$$

$$M\,dx + N\,dy = (x^2 - \cosh 1)\,dx \quad [\because y=1, dy=0]$$

$$\int_{BC} M\,dx + N\,dy = \int_{BC} (x^2 - \cosh 1)\,dx$$

$$= \int_{x=\pi}^{x=0} (x^2 - \cosh 1)\,dx$$

$$= \left[\frac{x^3}{3} - x\cosh 1\right]_{x=\pi}^{x=0}$$

$$\int_{BC} M\,dx + N\,dy = \pi\cosh 1 - \frac{\pi^3}{3} \quad \text{—— (7)}$$

<u>Case (iv)</u> <u>To evaluate</u> $\underline{\int_{CO} M\,dx + N\,dy}$ <u>(OR) Along the line CO</u> :-

We have $C(0,1)$ $O(0,0)$

Here $x=0 \Rightarrow dx=0$

y varies from 1 to 0

$\therefore$ y limits $y=1, y=0$.

$$Mdx + Ndy = (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$$

$$Mdx + Ndy = y\,dy$$

$$\int_{CO} Mdx + Ndy = \int_{CO} y\,dy$$

$$= \int_{y=1}^{y=0} y\,dy$$

$$= \left[\frac{y^2}{2}\right]_{y=1}^{y=0}$$

$$\int_{CO} Mdx + Ndy = -\frac{1}{2} \quad \text{--- (8)}$$

sub (5) (6) (7) and (8) in (4), we get

$$\oint_C Mdx + Ndy = \frac{\pi^3}{3} - \pi\cosh 0 + \frac{1}{2} + \pi\cosh 1 - \frac{\pi^3}{3} \quad [\because \cosh 0 = 1]$$

$$\oint_C Mdx + Ndy = \pi(\cosh 1 - 1)$$

$$\therefore \oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

∴ Green's theorem verified.

→ Applying Green's theorem to evaluate $\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where C is the boundary of the area enclosed by the x-axis and upper half of circle $x^2+y^2=a^2$

Sol: Given that $I = \oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ —— ①

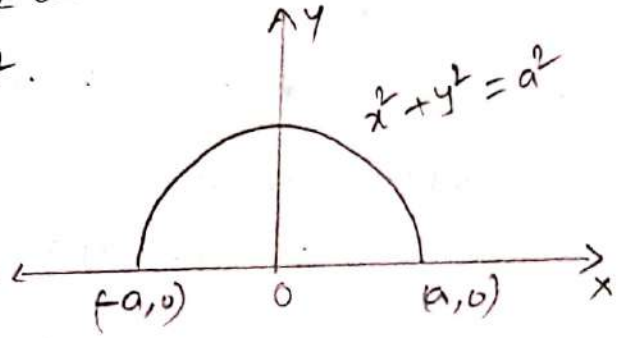
Wkt Green's Theorem in a plane.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—— ②}$$

Compare ① with L.H.S of ②, we get

Here $M = 2x^2-y^2 \quad N = x^2+y^2$

Given that c is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$.



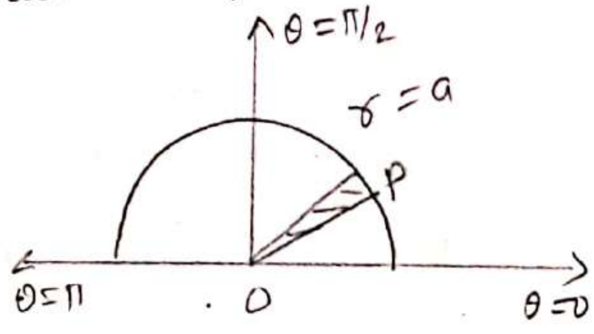
To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$:-

$M = 2x^2-y^2 \qquad N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x+y)$

To change ~~the order of~~ into polar coordinates put $x = r\cos\theta$, $y = r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$.



In the region θ varies from 0 to π

$\therefore$ θ limits $\theta=0,\ \theta=\pi$

Draw a radius vector OP in the region which starts at O $(r=0)$ and terminates at P (which is on the circle $r=a$)

$\therefore$ r limits $r=0,\ r=a$.

$\left[\because x^2+y^2=a^2 \quad r^2(\cos^2\theta+\sin^2\theta)=a^2 \quad r=a.\right.$

$$\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dx = \iint_R 2(x+y)\,dx\,dy$$

$$= 2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a} r(\cos\theta+\sin\theta)\,r\,dr\,d\theta$$

$\left[\because x = r\cos\theta,\ y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta\right.$

$$= 2\int_{r=0}^{r=a} r^2\,dr \int_{\theta=0}^{\theta=\pi} (\cos\theta + \sin\theta)\,d\theta$$

$$= 2\left[\frac{r^3}{3}\right]_{r=0}^{r=a} \left[\sin\theta - \cos\theta\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{2a^3}{3}\left[(\sin\pi - \cos\pi) - (\sin 0 - \cos 0)\right]$$

$$\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy = \frac{4a^3}{3}$$

→ Applying Green's theorem to evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where c is the square formed by the lines $x = \pm 1$ and $y \pm 1$.

sol: Given that $I = \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ —— ①

WKt Green's theorem.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \text{ —— ②}$$

Compare ① with L.H.S of ②,

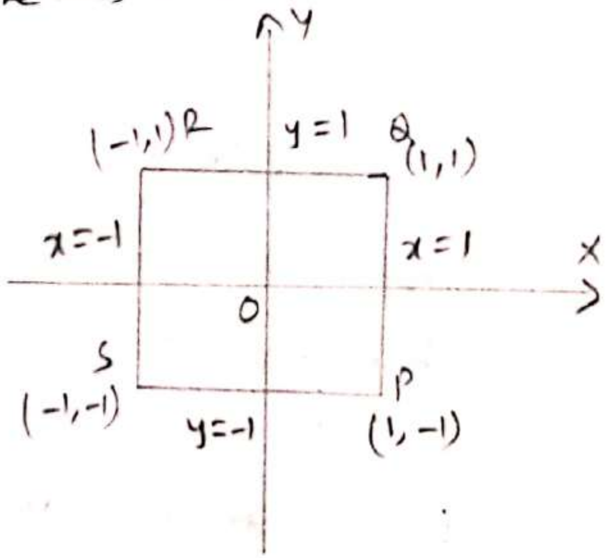
Here $M = x^2+xy$ $N = x^2+y^2$

$\frac{\partial M}{\partial y} = x$ $\frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x.$

Given that c is the square formed by the lines $x = \pm 1$, $y = \pm 1$

To evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$:-



In the region x varies from -1 to 1

∴ x limits are $x = -1$, $x = 1$.

y varies from -1 to 1

∴ y limits are $y = -1$, $y = 1$

$$\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \iint_R x\,dx\,dy$$

$$\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{x=-1}^{x=1}\int_{y=-1}^{y=1} x\,dx\,dy$$

$$= \int_{x=-1}^{x=1} x\,dx \int_{y=-1}^{y=1} dy$$

$$= \left[\frac{x^2}{2}\right]_{x=-1}^{x=1} \left[y\right]_{y=-1}^{y=1}$$

$$= \left(\frac{1}{2}-\frac{1}{2}\right)\left[1-(-1)\right]$$

$$\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy = 0$$

→ Applying Green's theorem to evaluate $\int_C e^x \sin y\,dx + e^x \cos y\,dy$. Where c is the rectangle whose vertices are $(0,0)$ $(1,0)$ $(1,\frac{\pi}{2})$ $(0,\frac{\pi}{2})$.

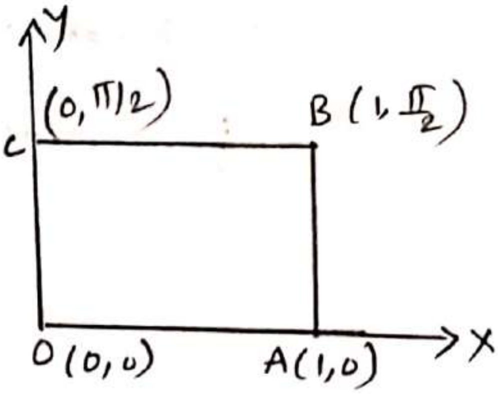
Sol:- Given that $I = \int_C e^x \sin y\,dx + e^x \cos y\,dy$ —①

Wkt Greens theorem. $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ —②

compare ① with L.H.S of ②, we get

$M = e^x \sin y \quad N = e^x \cos y$

The vertices of rectangle are $O(0,0)$ $A(1,0)$ $B(1,\frac{\pi}{2})$ $C(0,\frac{\pi}{2})$.



$$\frac{\partial M}{\partial y} = e^x \cos y \quad \frac{\partial N}{\partial x} = e^x \cos y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C (e^x \sin y)\,dx + (e^x \cos y)\,dy = \iint_R 0 . \,dx\,dy$$

$$\oint_C (e^x \sin y\,dx) + (e^x \cos y)\,dy = 0.$$

→ Use Green's theorem to evaluate $\oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy$ where C is the square bounded by $x=\pm 1$ and $y=\pm 1$.

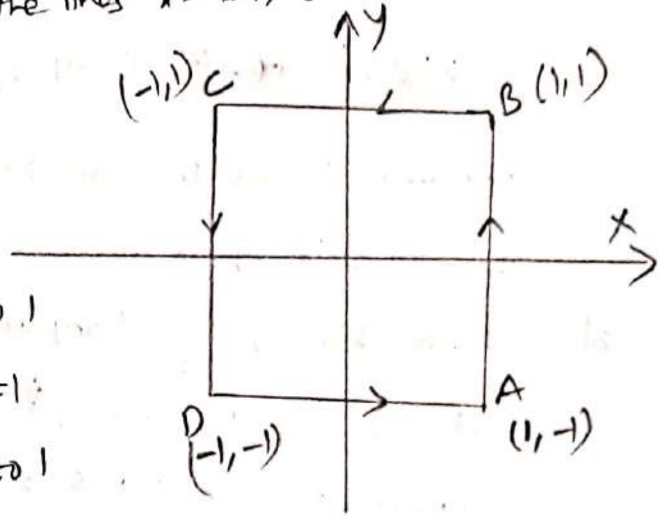
Sol:- Given that $I = \oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy$ — ①

Wkt Green's theorem $\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ — ②

Compare ① with L.H.S ② we get

$M = x^2(1+y)$ $N = x^3+y^3$.

Given that the square bounded by the lines $x=\pm 1$, $y=\pm 1$.



$\frac{\partial M}{\partial y} = x^2$ $\frac{\partial N}{\partial x} = 3x^2$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x^2$

In the region x varies from -1 to 1

∴ x limits $x=-1$, $x=1$.

y varies from -1 to 1

∴ y limits $y=-1$, $y=1$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy = \iint_R 2x^2\,dx\,dy$$

$$= \int_{x=-1}^{x=1}\int_{y=-1}^{y=1} 2x^2\,dx\,dy$$

$$= 2\int_{x=-1}^{x=1} x^2\,dx \int_{y=-1}^{y=1} dy$$

$$= 2\left[\frac{x^3}{3}\right]_{x=-1}^{x=1} \left[y\right]_{y=-1}^{y=1}$$

$$= \frac{2}{3}\left[1-(-1)\right]\left[1-(-1)\right]$$

$$\oint_C x^2(1+y)\,dx + (x^3+y^3)\,dy = \frac{8}{3}$$

Note :- Area of the plane region R bounded by a simple closed curve c.

WKT Green's theorem.

$$\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Let $M = -y$ $N = x$.

$$\oint_c x dy - y dx = \iint_R (1+1)\, dx\, dy$$

$$\oint_c x dy - y dx = 2\iint_R dx\, dy.$$

$$\oint_c x dy - y dx = 2\,(\text{Area of the region})$$

$$\therefore \text{Area} = \frac{1}{2}\oint_c x dy - y dx.$$

→ Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using Green's theorem.

Sol:- Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Parametric equations of ellipse are $x = a\cos\theta$ $y = b\sin\theta$.

$dx = -a\sin\theta\, d\theta$ $dy = b\cos\theta\, d\theta$.

$$\text{Area } A = \frac{1}{2}\oint_c x dy - y dx.$$

θ varies from 0 to 2π

$\therefore$ θ limits $\theta = 0$, $\theta = 2\pi$

$$A = \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} [a\cos\theta(b\cos\theta) - b\sin\theta(-a\sin\theta)]\, d\theta$$

$$= \frac{1}{2}ab\int_{\theta=0}^{\theta=2\pi} (\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= \frac{1}{2}ab\int_{\theta=0}^{\theta=2\pi} d\theta.$$

$$= \frac{1}{2}ab.[\theta]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{1}{2}ab(2\pi - 0)$$

$$\text{Area} = \pi ab.$$

→ Find the area of the circle $x^2+y^2=r^2$ using Green's Theorem.

Sol: Given that $x^2+y^2=r^2$.

Parametric equations of circle are $x=r\cos\theta$ $y=r\sin\theta$.

$dx=-r\sin\theta\,d\theta$, $dy=r\cos\theta\,d\theta$.

θ varies from 0 to 2π.

$\therefore$ θ limits $\theta=0$, $\theta=2\pi$.

Area $A=\frac{1}{2}\oint_C x\,dy-y\,dx$.

$$A=\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}[r\cos\theta\cdot r\cos\theta-r\sin\theta(-r\sin\theta)]\,d\theta$$

$$=\frac{1}{2}\int_{\theta=0}^{\theta=2\pi}r^2[\cos^2\theta+\sin^2\theta]\,d\theta$$

$$=\frac{r^2}{2}\int_{\theta=0}^{\theta=2\pi}d\theta$$

$$=\frac{r^2}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

$$=\frac{r^2}{2}[2\pi-0]$$

Area $A=\pi r^2$

Using Green's theorem, evaluate $\int_C xy^2\,dy - x^2y\,dx$. where c is the cardioid $r = a(1-\cos\theta)$.

sol:- We know that the Green's theorem.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_C xy^2\,dy - x^2y\,dx = \iint_R \left\{\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(-x^2y)\right\} dx\,dy$$

$$= \iint_R (x^2+y^2)\,dx\,dy$$ Where R is bounded by the cardioid $r = a(1-\cos\theta)$

To change it into polar co ordinates put $x = r\cos\theta$
$y = r\sin\theta$. $dx\,dy = r\,dr\,d\theta$.

In the region R, θ varies from $\theta = 0$ to $\theta = 2\pi$
r varies from $r = 0$ to $r = a(1-\cos\theta)$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a(1-\cos\theta)} r^2\, r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a(1-\cos\theta)} r^3\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^4}{4}\right]_{r=0}^{r=a(1-\cos\theta)} d\theta$$

$$= \frac{a^4}{4}\int_{\theta=0}^{\theta=2\pi} (1-\cos\theta)^4\,d\theta = \frac{a^4}{4}\int_{\theta=0} \left(2\sin^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 4a^4 \int_{\theta=0}^{\theta=2\pi} \sin^8\left(\frac{\theta}{2}\right) d\theta$$

Put $\frac{\theta}{2} = t$
$d\theta = 2dt$
when $\theta = 0, t = 0$
when $\theta = 2\pi, t = \pi$

$$= 4a^4 \int_{t=0}^{t\pi} \sin^8(t)\,2dt$$

$$= 16a^4 \int_{t=0}^{t=\pi} \sin^8(t)\,dt = 16a^4 \cdot \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{35}{16}\cdot\pi a^4.$$

Surface Integrals :-

A surface $\bar{r} = \bar{f}(u,v)$ is called a smooth surface if $\bar{f}(u,v)$ is continuous and possesses partial derivatives. Let $\bar{F}(\bar{r})$ be a continuous vector point function, defined over the smooth surface $\bar{r} = \bar{f}(u,v)$. Let S be the region of the surface. Divide the region into m subregions of areas $\delta S_1, \delta S_2, \delta S_3 \ldots \delta S_m$. Let P_i be a point of δS_i and $\bar{n}$ be the unit normal to δS_i at P_i. Let δA_i be the vector area of δS_i. Then $\delta\bar{A}_i = \bar{n}_i\,\delta S_i$

Form the sum $I_m = \sum_{i=1}^{m} \bar{F}(\bar{r}_i)\,\delta A_i = \sum \bar{F}(\bar{r}_i)\cdot\bar{n}_i\,dS_i$

Let m tend to infinity in such a way that each δS_i shrinks to a point. The limit of I_m if it exists is called the normal surface integral of $\bar{F}(\bar{r})$ over the region S of the surface $\bar{r} = \bar{f}(u,v)$ and is denoted by

$$\int_S \bar{F}(\bar{r})\,d\bar{A} \quad (\text{or}) \quad \int_S \bar{F}\cdot\bar{n}\,ds.$$

Note:- Other types surface integrals are $\int \bar{F}\times d\bar{A}$ (or) $\int \phi\, d\bar{A}$

Any integral which is to be evaluated over a surface is called a surface integral.

Surface Integrals - Cartesian Form :-

Let $\bar{F}(\bar{r}) = F_1 i + F_2 j + F_3 k$ where F_1, F_2, F_3 are continuous and differentiable functions of x, y, z.

Then $\int_S \bar{F}\cdot\bar{n}\,ds = \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy.$

Note:- Let R_1 be the projection of S on xy-plane Then

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

Let R_2 be the projection of S on yz-plane Then.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot i|}$$

Let R_3 be the projection of S on zx-plane Then

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \bar{F}\cdot\bar{n}\,\frac{dx\,dz}{|n\cdot j|}$$

→ Evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$ where $\bar{F} = 18z i - 12 j + 3y \bar{k}$ and S is the ~~surface of~~ part of the surface of the plane $2x+3y+6z=12$ located in the first octant.

Sol: Given that $\bar{F} = 18z i - 12 j + 3y k$.

We have to find $\int_S \bar{F}\cdot\bar{n}\, ds$

Given that the plane $\phi = 2x+3y+6z-12$.

Normal to the plane ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\nabla\phi = 2i+3j+6k.$$

Unit normal to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{2i+3j+6k}{\sqrt{2^2+3^2+6^2}} = \frac{2i+3j+6k}{7}.$$

Let R be the projection of S on xy-plane Then

$$\int_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

Given $\bar{F} = 18z i - 12 j + 3y k$

$$\bar{F}\cdot\bar{n} = (18z i - 12 j + 3y k)\cdot\left(\frac{2i+3j+6k}{7}\right)$$

$$\bar{F}\cdot\bar{n} = \frac{36z-36+18y}{7} = \frac{6}{7}(6z-6+3y)$$

$$\bar{n}\cdot k = \frac{1}{7}(2\bar{i}+3\bar{j}+6k)\cdot\bar{k} = \frac{6}{7}$$

Given surface is $2x+3y+6z=12$.

∴ Projection R of the plane $\phi = 2x+3y+6z-12=0$ in xy-plane is $2x+3y=12$

[∵ In xy-plane $z=0$]

$$\Rightarrow y = \frac{12-2x}{3}.$$

when $y=0 \Rightarrow x=6$.

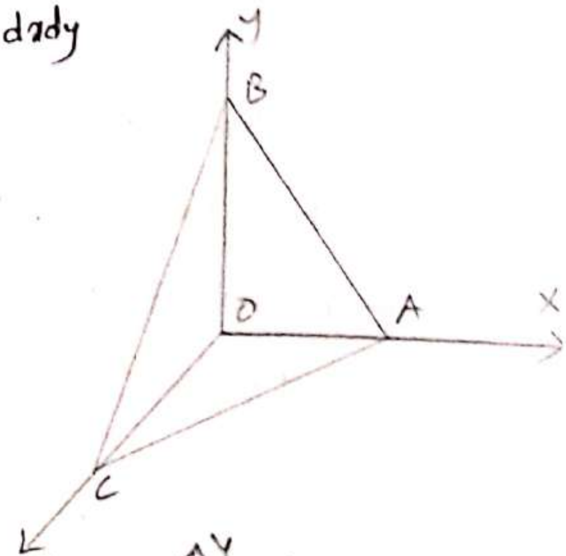
Now x varies from 0 to 6 and y varies from 0 to $\frac{12-2x}{3}$.

$$\int_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint_R \frac{6}{7}(6z-6+3y)\cdot\frac{dx\,dy}{\frac{6}{7}}$$

$$\int_S \vec{F}\cdot\hat{n}\, ds = \iint_R (6z-6+3y)\,dx\,dy$$

$$= \iint_R (12-2x-3y-6+3y)\,dx\,dy$$

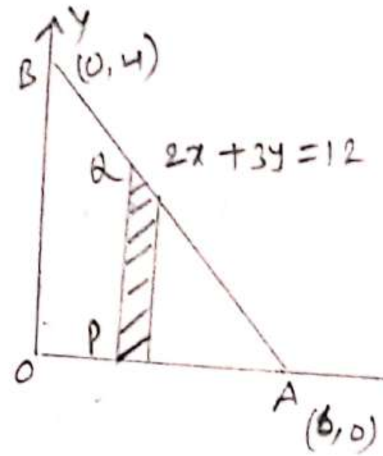


$$= \iint_R (6-2x)\,dx\,dy$$

$$= 2\iint_R (3-2x)\,dx\,dy$$

$$= 2\int_{x=0}^{x=6}\int_{y=0}^{y=\frac{12-2x}{3}} (3-x)\,dy\,.\,dx.$$

$$= 2\int_{x=0}^{x=6} (3-x)\,\left[y\right]_{y=0}^{y=\frac{12-2x}{3}} dx.$$



$$= 2\int_{x=0}^{x=6} (3-x)\cdot\frac{1}{3}(12-2x)\,dx.$$

$$= \frac{4}{3}\int_{x=0}^{x=6} (3-x)(6-x)\,dx.$$

$$= \frac{4}{3}\int_{x=0}^{x=6} (18-9x+x^2)\,dx = \frac{4}{3}\left[18x-\frac{9}{2}x^2+\frac{x^3}{3}\right]_{x=0}^{x=6}$$

$$= \frac{4}{3}\left[18.(6)-\frac{9}{2}(36)+\frac{6^3}{3}\right]$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = 24$$

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$\rightarrow$ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 12x^2y\,i - 3yz\,j + 2z\,k$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

Ans:- $\frac{-55}{24}$

sol:- Given that $\bar{F} = 12x^2y\,i - 3yz\,j + 2z\,k$.

We have to find $\int_S \bar{F}\cdot\bar{n}\,ds$

Given that the plane $\phi = x+y+z-1 = 0$.

Normal to the plane ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\nabla\phi = i+j+k.$$

Unit normal to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{i+j+k}{\sqrt{1^2+1^2+1^2}} = \frac{i+j+k}{\sqrt{3}}$$

Let R be the projection of S on xy-plane Then.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

Given $\bar{F} = 12x^2y\,i - 3yz\,j + 2z\,k$.

$$\bar{F}\cdot\bar{n} = (12x^2y\,i - 3yz\,j + 2z\,k)\cdot\frac{i+j+k}{\sqrt{3}}$$

$$= \frac{12x^2y - 3yz + 2z}{\sqrt{3}}$$

$$\bar{F}\cdot\bar{n} = \frac{1}{\sqrt{3}}\left[12x^2y - 3y(1-x-y) + 2(1-x-y)\right]$$

$$\bar{F}\cdot\bar{n} = \frac{1}{\sqrt{3}}\left[12x^2y + 3xy + 3y^2 - 2x - 5y + 2\right]$$

$$\bar{n}\cdot\bar{k} = \left(\frac{i+j+k}{\sqrt{3}}\right)\cdot k = \frac{1}{\sqrt{3}}$$

Given that plane $x+y+z-1=0$.

$\therefore$ Projection R of the plane $\phi = x+y+z-1=0$ is in xy-plane is $x+y=1$.

$$\Rightarrow y = 1-x.$$

when $y=0$, $x=1$.

Now x varies from 0 to 1 and y varies from 0 to $1-x$.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint_R \frac{(12x^2y+3xy+3y^2-2x-5y+2)}{\sqrt{3}}\cdot\frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\left[12x^2y+3xy+3y^2-2x-5y+2\right]dy\,dx$$

$$= \int_{x=0}^{x=1}\left[12x^2\cdot\frac{y^2}{2}+3x\cdot\frac{y^2}{2}+3\cdot\frac{y^3}{3}-2xy-5\frac{y^2}{2}+2y\right]_{y=0}^{y=1-x}dx$$

$$= \int_{x=0}^{x=1}\left[\left(6x^2+\frac{3}{2}x+\frac{5}{2}\right)y^2+y^3-2xy+2y\right]_{y=0}^{y=1-x}dx\,.$$

$$= \int_{x=0}^{x=1}\left[\left(6x^2+\frac{3}{2}x-\frac{5}{2}\right)(1-x)^2+(1-x)^3-2x(1-x)+2(1-x)\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{x=1}(x^3+11x^2-x-8)\,dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}+11\frac{x^3}{3}-\frac{x^2}{2}-8x\right]_{x=0}^{x=1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}+\frac{11}{3}-\frac{1}{2}-8\right)-0\right]$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \frac{-55}{24}$$

→ Evaluate $\int \bar{F}\cdot\bar{n}\,ds$ where $F = zi + xj - 3y^2zk$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol:- Given that $\bar{F} = zi + xj - 3y^2zk$.

We have to find $\int \bar{F}\cdot\bar{n}\,ds$.

Given that the surface S is $\phi = x^2+y^2-16$

Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\nabla\phi = 2xi + 2yj$$

Unit normal to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}} = \frac{xi+yj}{\sqrt{16}}$$

$$\bar{n} = \frac{xi+yj}{4}.$$

Let R be the projection of S on yz plane. Then R is the rectangle OBCD.

$$\iint_S F\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$$

$$\bar{F}\cdot\bar{n} = (zi+xj-3y^2zk)\cdot\frac{(xi+yj)}{4} = \frac{xz+xy}{4}$$

$$\bar{n}\cdot i = \frac{1}{4}(xi+yj)\cdot i = \frac{x}{4}.$$

For the surface $x^2+y^2=16$ in the yz plane, $x=0 \Rightarrow y=4$.

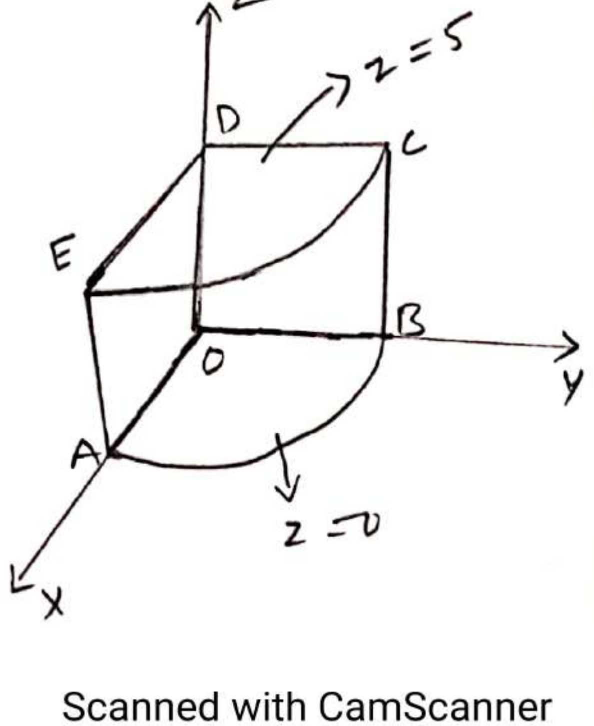
Hence in the first octant y varies from 0 to 4, z varies from 0 to 5.

Then $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$

$$= \iint_R \frac{x(y+z)}{4}\cdot\frac{dy\,dz}{\frac{x}{4}}$$

$$= \int_{y=0}^{y=4}\int_{z=0}^{z=5}(y+z)\,dz\cdot dy$$

$$= \int_{y=0}^{y=4}\left[yz+\frac{z^2}{2}\right]_{z=0}^{z=5} dy$$



$$\iint_S F\cdot\bar{n}\,ds = \int_{y=0}^{y=4}\left(5y+\frac{25}{2}\right)dy$$

$$=\left[\frac{5}{2}y^2+\frac{25}{2}y\right]_{y=0}^{y=4}$$

$$=\left(\frac{5}{2}\cdot 16+\frac{25}{2}\cdot 4\right)-0$$

$$=(40+50)$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = 90.$$

→ Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ if $\bar{F} = yz\,i + 2y^2 j + xz^2 k$ and s is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ and $z=2$

Ans:- 78

Sol:- Given that $\bar{F} = yz\,i + 2y^2 j + xz^2 k$.

We have to find $\int_S \bar{F}\cdot\bar{n}\,ds$

Given that the surface S is $\phi = x^2+y^2-9$

Normal to the surface ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$\nabla\phi = 2xi + 2yj$$

Unit normal to the surface ϕ is $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\bar{n} = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}} = \frac{xi+yj}{3}$$

Let R be the projection of S on YZ plane Then R is the rectangle OBCD.

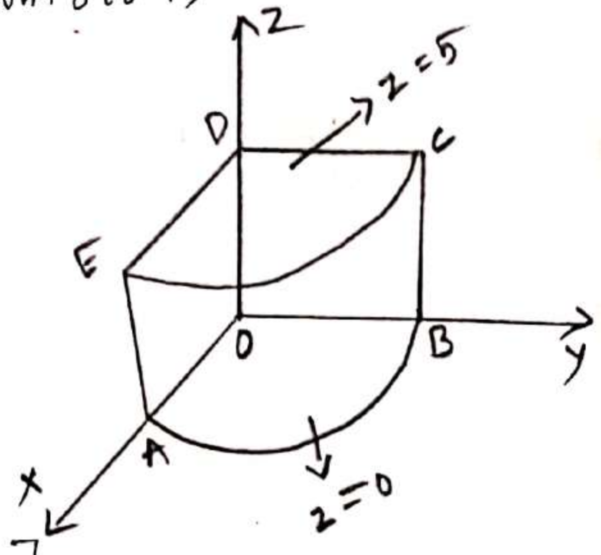
$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$$

$$\bar{F}\cdot\bar{n} = (yz\,i + 2y^2 j + xz^2 k)\cdot\left(\frac{xi+yj}{3}\right) = \frac{xyz+2y^3}{3}$$

$$n\cdot i = \left(\frac{xi+yj}{3}\right)\cdot i = \frac{x}{3}.$$

For the surface $x^2+y^2=9$ in the YZ plane $x=0 \Rightarrow y=3$.

Hence in the first octanct y varies from 0 to 4, z varies from 0 to 5



Then $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$

$$= \iint_R \frac{xyz+2y^3}{3}\cdot\frac{dy\,dz}{\frac{x}{3}}$$

$$= \iint_R \left(yz + \frac{2y^3}{x}\right)dy\,dz$$

$$= \int_{y=0}^{y=3}\int_{z=0}^{z=5}\left[yz + \frac{2y^3}{\sqrt{9-y^2}}\right]dy\,dz$$

$$\iint_S \bar{F}.\bar{n}\,ds = \int_{y=0}^{y=3}\left[y\cdot\frac{z^2}{2}+\frac{2y^3}{\sqrt{9-y^2}}\,z\right]_{z=0}^{z=2} dy$$

$$= \int_{y=0}^{y=3}\left[2y+\frac{4y^3}{\sqrt{9-y^2}}\right] dy$$

$$= \int_{y=0}^{y=3} 2y\,dy + 4\int_{y=0}^{y=3}\frac{y^3}{\sqrt{9-y^2}}\,dy$$

$$= \left[2\cdot\frac{y^2}{2}\right]_{y=0}^{y=3} + 4\int_{\theta=0}^{\theta=\pi/2}\frac{27\sin^3\theta\;3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

Put $y = 3\sin\theta$

$dy = 3\cos\theta\,d\theta$

when $y=0$, $\theta=0$

when $y=3$ $\theta=\frac{\pi}{2}$

$$= (9-0) + 4\cdot 27\int_{\theta=0}^{\theta=\pi/2}\sin^3\theta\,d\theta$$

$$= 9 + 4\cdot 27\cdot\int_{\theta=0}^{\theta=\pi/2}\frac{3\sin\theta-\sin3\theta}{4}\,d\theta$$

$$= 9+27\int_{\theta=0}^{\theta=\pi/2}(3\sin\theta-\sin3\theta)\,d\theta$$

$$= 9+27\left[\frac{\cos3\theta}{3}-3\cos\theta\right]_{\theta=0}^{\theta=\pi/2}$$

$$= 9+27\left[0-\left(\frac{1}{3}-3\right)\right]$$

$$= 9+27\cdot\frac{8}{3}$$

$$\iint_S \bar{F}.\bar{n}\,ds = 80$$

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Stoke's Theorem :- (Transformation between Line Integral and Surface Integral)

Let s be a open surface bounded by a closed, non intersecting curve c. If $\bar{F}$ is any differentiable vector point function then $\oint_c \bar{F}.d\bar{r} = \int_s \text{curl}\,\bar{F}.\bar{n}\,ds$

Where c is traversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

Deduction of Green's Theorem from Stoke's Theorem :-

Let the surface lie on xy plane. Then z-axis will be along the normal $\bar{n}$ i.e $\bar{n} = \bar{k}$

Let $\bar{F} = F_1\bar{i} + F_2\bar{j}$, $\bar{r} = x\bar{i} + y\bar{j}$

$$\therefore \bar{F}.d\bar{r} = F_1\,dx + F_2\,dy$$

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix}$$

$$= \frac{\partial F_2}{\partial z}\bar{i} - \frac{\partial F_1}{\partial z}\bar{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

$$\therefore (\nabla\times\bar{F}).\bar{n} = (\nabla\times\bar{F}).\bar{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

In xy-plane $ds = dx\,dy$

$$\therefore \oint_c \bar{F}.d\bar{r} = \int (\nabla\times\bar{F}).\bar{n}\,ds$$

$$\int_c F_1\,dx + F_2\,dy = \iint_s \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial y}\right) dx\,dy$$

This is same as Green's theorem in a plane.

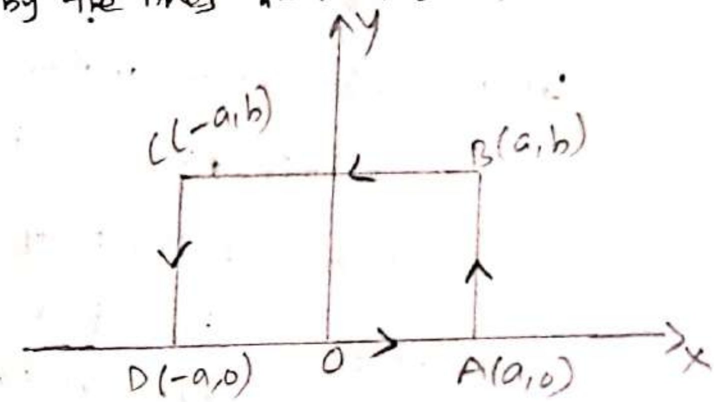
→ Verify stoke's theorem for $\bar{F} = (x^2+y^2)i - 2xy\,j$ taken round the rectangle bounded by the lines $x = \pm a$, $y=0, y=b$.

Sol:- Given that $\bar{F} = (x^2+y^2)i - 2xy\,j$

WKT stoke's theorem

$$\oint_C \bar{F}.d\bar{r} = \int_S \text{curl}\bar{F}.\bar{n}\,ds$$

Given that the rectangle bounded by the lines $x = \pm a$, $y=0, y=b$.



To evaluate $\int_S \text{curl}\bar{F}.\bar{n}\,ds$:-

$$\bar{F} = (x^2+y^2)i - 2xy\,j$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$

$F_1 = x^2+y^2$ $F_2 = -2xy$ $F_3 = 0$.

$$\text{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2xy)\right) - j\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+y^2)\right) + k\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2+y^2)\right)$$

$$= k(-2y-2y)$$

$$\text{curl}\bar{F} = -4yk.$$

The rectangle ABCD is in xy-plane. z-axis is perpendicular to xy-plane.

Along z-axis $\bar{k}$ is the unit normal vector.

so $\bar{n} = \bar{k}$

$$\text{curl}\bar{F}.\bar{n} = (-4y\bar{k}).\bar{k} = -4y$$

In the region x varies from $-a$ to a

$\therefore$ x limits are $x=-a, x=a$.

y varies from 0 to b

$\therefore$ y limits are $y=0, y=b$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_S -4y\,dx\,dy \qquad \left[\because \text{Rectangle is in } xy\text{-plane. } ds = dx\,dy\right]$$

$$= \int_{x=-a}^{x=a}\int_{y=0}^{y=b} -4y\,dx\,dy$$

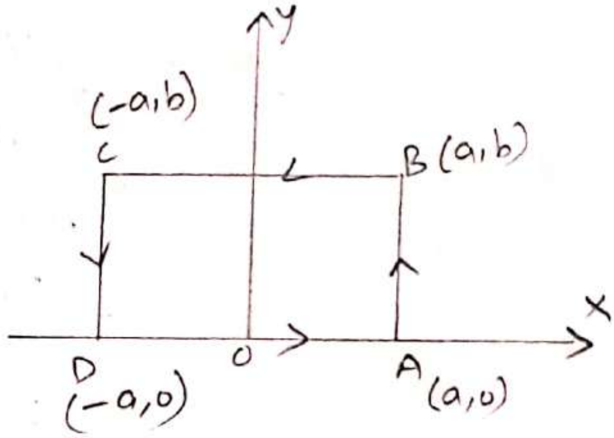
$$= \int_{x=-a}^{x=a} dx \int_{y=0}^{y=b} -4y\,dy$$

$$= [x]_{x=a}^{x=a}\left[-4\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$[a-(-a)]\left[-2b^2-0\right]$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = -4ab^2$$

To evaluate $\oint_C \bar{F}\cdot d\bar{r}$:-



We have $\bar{F} = (x^2+y^2)i - 2xy\,j$

$\bar{r} = xi + yj \quad (\because xy\text{-plane})$

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = \left[(x^2+y^2)i - 2xy\,j\right]\cdot\left[dx\,i + dy\,j\right]$

$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx - 2xy\,dy$ —— (1)

To evaluate the line integral $\oint_C \bar{F}\cdot d\bar{r}$

We can write $\oint_C \bar{F}\cdot d\bar{r} = \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CD}\bar{F}\cdot d\bar{r} + \int_{DA}\bar{F}\cdot d\bar{r}$ —— (2)

(i) To evaluate $\int_{AB}\bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have A (a,0) B (a,b)

$x = a \Rightarrow dx = 0$

y varies from 0 to b.

$\therefore$ y limits $y=0, y=b$

From(1) $\bar{F}\cdot d\bar{r} = -2ay\,dy \quad [\because x=a,\ dx=0]$

$$\int_{AB}\bar{F}\cdot d\bar{r} = \int_{AB} -2ay\,dy$$

$$= \int_{y=0}^{y=b} -2ay\,dy$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \left[-2a\cdot\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = -ab^2 \quad \text{--- (3)}$$

(ii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(a,b)$ $C(-a,b)$

Here $y=b \Rightarrow dy=0$

x varies from $+a$ to $-a$

$\therefore$ x limits are $x=+a, x=-a$

From (1) $\bar{F}\cdot d\bar{r} = (x^2+b^2)\,dx$ $\left[\because y=b,\ dy=0\right]$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2+b^2)\,dx$$

$$= \int_{x=+a}^{x=-a} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3}+b^2x\right]_{x=+a}^{x=-a}$$

$$= \left(-\frac{a^3}{3}-ab^2\right)-\left(+\frac{a^3}{3}+ab^2\right)$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = -\frac{2a^3}{3}-2ab^2 \quad \text{--- (4)}$$

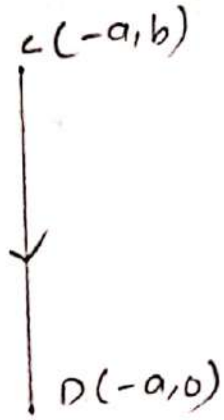
(iii) To evaluate $\int_{CD} \bar{F}\cdot d\bar{r}$ (OR) Along the line CD :-

We have $C(-a,b)$ $D(-a,0)$

Here $x=-a \Rightarrow dx=0$

y varies from b to 0

$\therefore$ y limits are $y=b,\ y=0$



From (1) $\bar{F}\cdot d\bar{r} = 2ay\,dy$ $\left[\because x=-a,\ dx=0\right]$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} 2ay\,dy$$

$$= \int_{y=b}^{y=0} 2ay\,dy$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \left[2a\frac{y^2}{2}\right]_{y=b}^{y=0}$$

$$= 0 - ab^2$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = -ab^2 \quad \text{——} (5)$$

(iv) To evaluate $\int_{DA} \bar{F}\cdot d\bar{r}$ (OR) Along the line DA :—

We have $D(-a,0)$ $A(a,0)$.

We have $y=0 \Rightarrow dy=0$.

x varies from $-a$ to a.

$\therefore$ x limits are $x=-a, x=a$.

From (1) $\bar{F}\cdot d\bar{r} = x^2\,dx$ $[\because y=0, dy=0]$

$$\int_{DA} \bar{F}\cdot d\bar{r} = \int_{DA} x^2\,dx$$

$$= \int_{x=-a}^{x=a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{x=-a}^{x=a}$$

$$= \frac{a^3}{3} - \frac{(-a)^3}{3}$$

$$\int_{DA} \bar{F}\cdot d\bar{r} = \frac{2a^3}{3} \quad \text{——} (6)$$

Sub. (3) (4) (5) and (6) in (2), we get

$$\oint_C \bar{F}\cdot d\bar{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$\oint_C \bar{F}\cdot d\bar{r} = -4ab^2 \quad \text{——} (7).$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$\therefore$ Stokes theorem verified

→ Verify stokes theorem for $\bar{F} = y^2 i - 2xy j$ taken round the rectangle bounded by $x = \pm b$, $y = 0$, $y = a$.

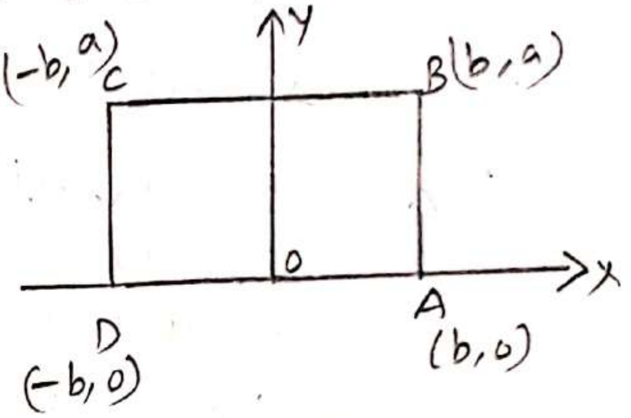
Ans:: $-4a^2b$.

Sol:- Given that $F = y^2 i - 2xy j$

Wkt stokes theorem

$$\oint_c \bar{F}.d\bar{r} = \int_s \text{curl}\bar{F}.\bar{n}\, ds$$

Given that the rectangle bounded by the lines $x = \pm b$, $y = 0$, $y = a$.



To evaluate $\int_s \text{curl}\bar{F}.\bar{n}\, ds$.

$$\bar{F} = y^2 i - 2xy j$$

We have, $\bar{F} = F_1 i + F_2 j + F_3 k$.

$F_1 = y^2 \quad F_2 = -2xy \quad F_3 = 0$,

$$\text{Curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix}$$

$$\text{Curl}\,\bar{F} = -4y\bar{k}$$

The rectangle ABCD is in xy-plane. z-axis is perpendicular to xy-plane.

Along z-axis $\bar{k}$ is the unit normal vector.

$$\text{so } \bar{n} = \bar{k}.$$

$$\text{Curl}\bar{F}.\bar{n} = (-4y\bar{k}).\bar{k} = -4y$$

In the region x varies from $-b$ to b

$\therefore$ x limits are $x = -b$, $x = b$.

y varies from 0 to a.

$\therefore$ y limits are $y = 0$, $y = a$.

Rectangle is in xy-plane $ds = dx\,dy$.

$$\int_s \text{curl}\bar{F}.\bar{n}\, ds = \int_s -4y\, dx\,dy$$

$$= \int_{x=-b}^{x=b} \int_{y=0}^{y=a} -4y\, dx\,dy$$

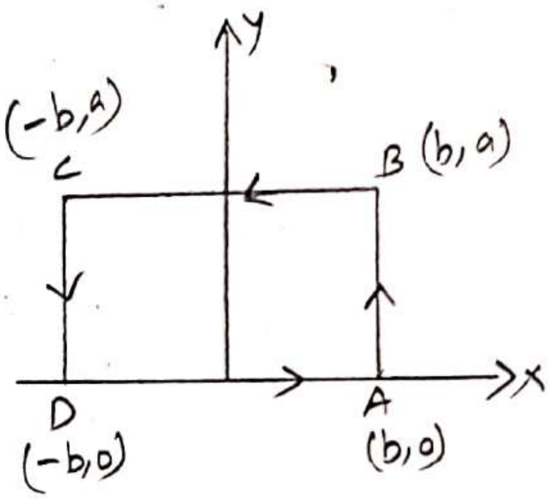
$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \int_{x=-b}^{x=b} dx \int_{y=0}^{y=a} -4y\,dy$$

$$= [x]_{x=-b}^{x=b}\left[-\frac{4y^2}{2}\right]_{y=0}^{y=a}.$$

$$= [b-(-b)](-2a^2-0)$$

$$\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = -4a^2b$$

To evaluate $\oint_C \bar{F}\cdot d\bar{r}$:-



We have $\bar{F} = y^2 i - 2xy\,j$

$\bar{r} = xi + yj$

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = [y^2 i - 2xy\,j]\cdot[dx\,i + dy\,j]$

$\bar{F}\cdot d\bar{r} = y^2dx - 2xy\,dy$ —— ①.

To evaluate the line integral $\oint_C \bar{F}\cdot d\bar{r}$

We can write $\oint_C \bar{F}\cdot d\bar{r} = \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CD}\bar{F}\cdot d\bar{r} + \int_{DA}\bar{F}\cdot d\bar{r}$ —— ②.

(i) To evaluate $\int_{AB}\bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(b,0)$ $B(b,a)$

Here $x=b \Rightarrow dx=0$.

y varies from 0 to a.

$\therefore$ y limits $y=0,\ y=a$.

From ① $\bar{F}\cdot d\bar{r} = -2by\,dy$.

$$\int_{AB}\bar{F}\cdot d\bar{r} = \int_{AB} -2by\,dy$$

$$= -2b\int_{y=0}^{y=a} y\,dy$$

$$= -2b\left[\frac{y^2}{2}\right]_{y=0}^{y=a}$$

$$\int_{AB}\bar{F}\cdot d\bar{r} = -a^2b \quad \text{——} ③.$$

(ii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :-

We have $B(b,a)$ $C(-b,a)$

Here $y=a \Rightarrow dy=0$.

x varies from b to $-b$

$\therefore$ x limits are $x=b,\ x=-b$.

From (1) $\bar{F}\cdot d\bar{r} = a^2\,dx$.

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} a^2\,dx$$

$$= a^2 \int_{x=b}^{x=-b} dx$$

$$= a^2 \left[x\right]_{x=b}^{x=-b}$$

$$= a^2(-b-b)$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = -2a^2 b \quad \text{—— (4)}$$

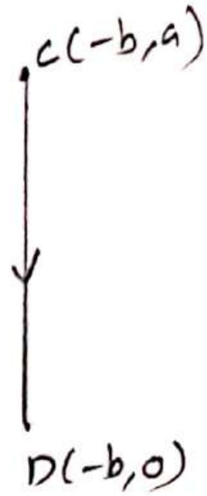
(iii) To evaluate $\int_{CD} \bar{F}\cdot d\bar{r}$ (or) Along the line CD :-

We have $C(-b,a)$ $D(-b,0)$

Here $x=-b \Rightarrow dx=0$.

y varies from a to 0.

$\therefore$ y limits are $y=a,\ y=0$.



From (1) $\bar{F}\cdot d\bar{r} = 2by\,dy$

$$\int_{CD} \bar{F}\cdot d\bar{r} = \int_{CD} 2by\,dy$$

$$= 2b \int_{y=a}^{y=0} y\,dy$$

$$= 2b \left[\frac{y^2}{2}\right]_{y=a}^{y=0}$$

$$\int_{CD} \bar{F}\cdot d\bar{r} = -a^2 b \quad \text{—— (5)}$$

(iv) To evaluate $\int_{DA} \bar{F}\cdot d\bar{r}$ (or) Along the line DA :-

Here $D(-b,0)$ $A(b,0)$.

$y=0 \Rightarrow dy=0$.

x varies from $-b$ to b.

$\therefore$ x limits $x=-b,\ x=b$.

From (1) $\bar{F}\cdot d\bar{r}=0$

$$\int_{DA} \bar{F}\, d\bar{r} = 0. \quad \text{——— (6)}$$

Sub. (3) (4) (5) and (6) in (2), we get

$$\oint_C \bar{F}\, d\bar{r} = -a^2b - 2a^2b - a^2b$$

$$\oint_C \bar{F}\cdot d\bar{r} = -4a^2b$$

$$\therefore \oint_C \bar{F}\, d\bar{r} = \int_S \text{curl}\, \bar{F}\cdot \bar{n}\, ds$$

$\therefore$ Stokes Theorem verified.

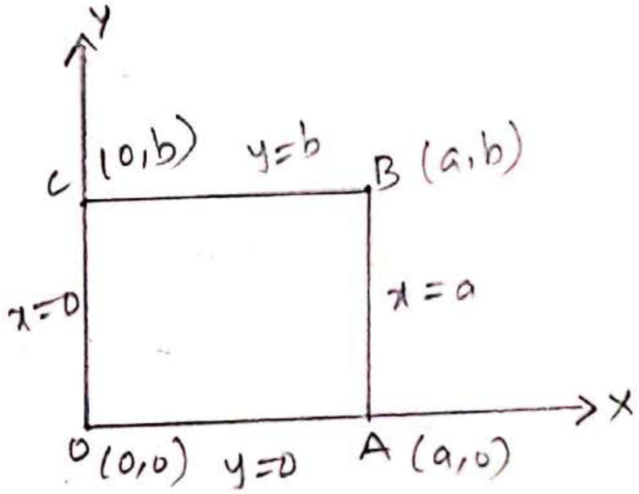
→ Verify Stoke's theorem for $\bar{F} = (x^2-y^2)i + 2xyj$ over the box bounded by the planes $x=0,\ x=a,\ y=0,\ y=b$.

Sol:- Given that $\bar{F} = (x^2-y^2)i + 2xyj$

WKT Stoke's theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds.$$

Given that the box bounded by the planes $x=0,\ x=a,\ y=0,\ y=b$.



To evaluate $\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$:-

$$\bar{F} = (x^2-y^2)i + 2xyj$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k.$

$F_1 = x^2-y^2 \quad F_2 = 2xy \quad F_3 = 0$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right) - j\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2-y^2)\right) + k\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2-y^2)\right)$$

$$= k(2y+2y)$$

$$\text{curl}\,\bar{F} = 4yk.$$

The rectangle OABC is in xy-plane. z-axis is perpendicular to xy-plane. Along z-axis, $\bar{k}$ is the unit normal vector.

$$\text{so } \bar{n} = \bar{k}$$

$$\text{curl}\,\bar{F}\cdot\bar{n} = (4y\bar{k})\cdot\bar{k} = 4y$$

In the region x varies from 0 to a.

$\therefore$ x limits are $x=0,\ x=a$.

y varies from 0 to b.

$\therefore$ y limits are $y=0,\ y=b$.

$$\int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = \int_S (4y\bar{k})\cdot\bar{k}\,dx\,dy \quad \left[\because \text{ Rectangle is in } xy\text{-plane } ds = dx\,dy\right]$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b} 4y\,dx\,dy$$

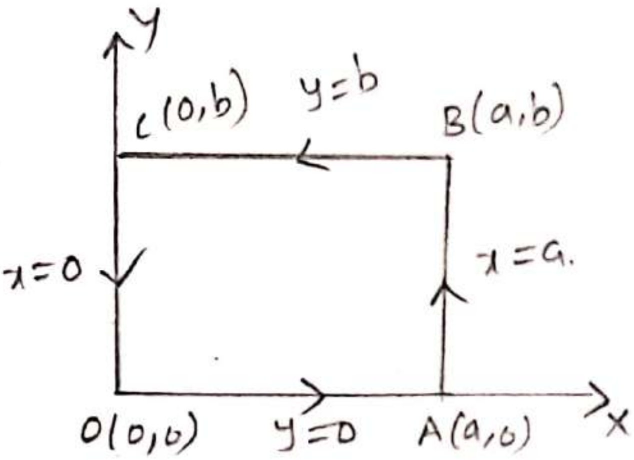
$$= 4\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} y\,dy$$

$$= 4\,[x]_{x=0}^{x=a}\left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$= 4(a-0)\left(\frac{b^2}{2}-0\right)$$

$$\int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = 2ab^2$$

To evaluate $\oint_C \bar{F}\cdot d\bar{r}$:-



We have $\bar{F} = (x^2+y^2)i + 2xy\,j$

$\bar{r} = xi + yj \quad (\because xy\text{-plane})$

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}\cdot d\bar{r} = \left[(x^2-y^2)i + 2xy\,j\right]\cdot\left[dx\,i + dy\,j\right]$

$\bar{F}\cdot d\bar{r} = (x^2-y^2)dx + 2xy\,dy$ —— (1)

To evaluate the line integral $\oint_C \bar{F}\cdot d\bar{r}$

We can write $\oint_C \bar{F}\cdot d\bar{r} = \int_{OA}\bar{F}\cdot d\bar{r} + \int_{AB}\bar{F}\cdot d\bar{r} + \int_{BC}\bar{F}\cdot d\bar{r} + \int_{CO}\bar{F}\cdot d\bar{r}$ —— (2).

Case (i) :- To evaluate $\int_{OA}\bar{F}\cdot d\bar{r}$ (or) Along the line OA :-

We have O(0,0) A(a,0)

Here $y=0 \Longrightarrow dy=0$.

x varies from 0 to a.

$\therefore$ x limits $x=0,\ x=a$.

From (1) $\bar{F}\cdot d\bar{r} = x^2\,dx$. $[\because y=0,\ dy=0]$

$$\int_{OA}\bar{F}\cdot d\bar{r} = \int_{OA} x^2\,dx$$

$$= \int_{x=0}^{x=a} x^2\,dx.$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \left[\frac{x^3}{3}\right]_{x=0}^{x=a}$$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \frac{a^3}{3} \quad \text{——} (3).$$

Case(ii) To evaluate $\int_{AB} \bar{F}\cdot d\bar{r}$ (or) Along the line AB :-

We have $A(a,0)$ $B(a,b)$

Here $x=a \Rightarrow dx=0$

y varies from 0 to b.

$\therefore$ y limits $y=0$ $y=b$.

From (1) $\bar{F}\cdot d\bar{r} = 2ay\, dy$. $[\because x=a, dx=0]$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} 2ay\, dy$$

$$= \int_{y=0}^{y=b} 2ay\, dy$$

$$= 2a.\left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$\int_{AB} \bar{F}\cdot d\bar{r} = ab^2 \quad \text{——} (4).$$

Case (iii) To evaluate $\int_{BC} \bar{F}\cdot d\bar{r}$ (or) Along the line BC :—

We have $B(a,b)$ $C(0,b)$.

Here $y=b \Rightarrow dy=0$.

x varies from a to 0.

$\therefore$ x limits $x=a$, $x=0$.

From (1) $\bar{F}\cdot d\bar{r} = (x^2-b^2)\, dx$ $[\because y=b, dy=0]$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} (x^2-b^2)\, dx$$

$$= \int_{x=a}^{x=0} (x^2-b^2)\, dx.$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \left[\frac{x^3}{3} - b^2 x\right]_{x=a}^{x=0}$$

$$= 0 - \left(\frac{a^3}{3} - ab^2\right)$$

$$\int_{BC} \bar{F}\cdot d\bar{r} = ab^2 - \frac{a^3}{3} \quad \text{—— (5)}$$

Case (iv) To evaluate $\int_{CO} \bar{F}\cdot d\bar{r}$ (or) Along the line CO :-

We have $C(0,b)$ $O(0,0)$

Here. $x=0 \Rightarrow dx=0$.

y varies from b to 0.

$\therefore$ y limits $y=b$, $y=0$.

From (1). $\bar{F}\cdot d\bar{r} = 0$ $\quad [\because x=0, dx=0]$.

$$\int_{CO} \bar{F}\cdot d\bar{r} = 0 \quad \text{—— (6)}$$

sub. (3), (4) (5) and (6) in (2), we get.

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0.$$

$$\oint_C \bar{F}\cdot d\bar{r} = 2ab^2.$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds.$$

$\therefore$ stokes theorem verified.

→ Verify Stoke's theorem for $\bar{F} = (2x-y)i - yz^2 j - y^2 z \bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Sol:- Given that $\bar{F} = (2x-y)i - yz^2 j - y^2 z k$.

WKT Stoke's theorem $\oint_C \bar{F}.d\bar{r} = \int_S \text{curl}\,\bar{F}.\bar{n}\,ds$.

Given that the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

<u>To evaluate $\int_S \text{curl}\bar{F}.\bar{n}\,ds$:-</u>

$$\bar{F} = (2x-y)i - yz^2 j - y^2 z k.$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$.

$$F_1 = 2x-y \quad F_2 = -yz^2 \quad F_3 = -y^2 z.$$

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(-y^2 z) - \frac{\partial}{\partial z}(-yz^2)\right) - j\left(\frac{\partial}{\partial x}(-y^2 z) - \frac{\partial}{\partial z}(2x-y)\right) + k\left(\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right)$$

$$= i(-2yz+2yz) - j(0+0) + k(0+1)$$

$$\text{curl}\bar{F} = \bar{k}$$

The region R is the projection of S on xy-plane.

z-axis is perpendicular xy-plane. so along z-axis unit normal vector $\bar{n} = \bar{k}$

$$\therefore \text{curl}\,\bar{F}.\bar{n} = \bar{k}.\bar{k} = 1.$$

$$ds = dx\,dy \quad [\because \text{In xy-plane}]$$

$\int_S \text{Curl}\,\bar{F}.\bar{n}\,ds = \int_S k.k\,dx\,dy = \iint_R dx\,dy.$

$=$ Area of the circle.

$\int_S \text{Curl}\,\bar{F}.\bar{n}\,ds = \pi.$

$\because x^2+y^2+z^2=1$

In xy-plane $z=0$

$x^2+y^2=1$

which is circle and radius $r=1$

To evaluate $\oint_C \bar{F}.d\bar{r}$:-

We have $\bar{F} = (2x-y)i - yz^2 j - y^2 z k.$

In xy plane $z=0$

$\bar{F} = (2x-y)i$

$\bar{r} = xi + yj$

$d\bar{r} = dx\,i + dy\,j$

$\bar{F}.d\bar{r} = (2x-y)i + 0j + 0k].[dx\,i + dy\,j]$

$\bar{F}.d\bar{r} = (2x-y)dx$

The boundary c of s is a circle in xy-plane i.e $x^2+y^2=1,\ z=0.$

The parametric equations of circle are $x=\cos\theta \quad y=\sin\theta$

$dx = -\sin\theta\,d\theta \quad dy = \cos\theta\,d\theta$

θ varies from 0 to 2π

θ limits $\theta=0,\ \theta=2\pi$.

$\bar{F}.d\bar{r} = (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$

$\bar{F}.d\bar{r} = (\sin^2\theta - \sin 2\theta)\,d\theta$

$\oint_C \bar{F}.d\bar{r} = \int_C (\sin^2\theta - \sin 2\theta)\,d\theta$

$= \int_{\theta=0}^{\theta=2\pi} \left[\frac{1-\cos 2\theta}{2} - \sin 2\theta\right] d\theta$

$= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} d\theta - \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} \cos 2\theta\,d\theta - \int_{\theta=0}^{\theta=2\pi} \sin 2\theta\,d\theta$

$= \frac{1}{2}[\theta]_{\theta=0}^{\theta=2\pi} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi} + \left[\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi}$

$$\oint_C \bar{F}\cdot d\bar{r} = \frac{1}{2}(2\pi - 0) - \frac{1}{2}\left[\frac{\sin 4\pi}{2} - 0\right] + \left[\frac{\cos 4\pi}{2} - \frac{\cos 0}{2}\right]$$

$$\oint_C \bar{F}\cdot d\bar{r} = \pi .$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\bar{F}\cdot\bar{n}\, ds$$

$\therefore$ stoke's theorem verified.

→ Apply stokes theorem, to evaluate $\oint_C (y\,dx + z\,dy + x\,dz)$ where c is the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and $x+z=a$.

Sol:- Given that $\oint_C (y\,dx + z\,dy + x\,dz)$

Let $\bar{F}\cdot d\bar{r} = y\,dx + z\,dy + x\,dz$

$\bar{F}\cdot d\bar{r} = (yi + zj + xk)\cdot(i\,dx + j\,dy + k\,dz)$

$\Rightarrow \bar{F} = yi + zj + xk \quad d\bar{r} = i\,dx + j\,dy + k\,dz.$

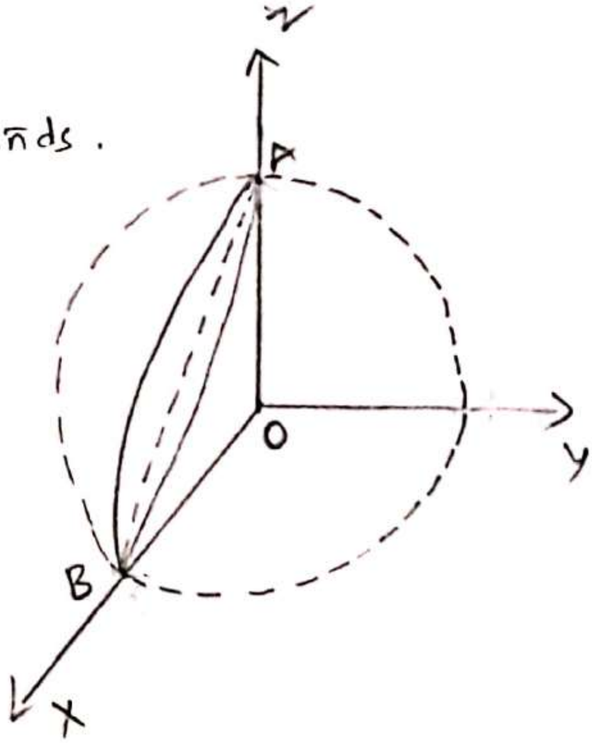
The intersection of the sphere $x^2+y^2+z^2=a^2$ and the plane $x+z=a$ is a circle in the plane $x+z=a$ with AB as diameter

Equation of the plane is $x+z=a \Rightarrow \frac{x}{a}+\frac{z}{a}=1.$

$\therefore OA = OB = a$ i.e $A(a,0,0)$ and $B(0,0,a)$.

$\therefore$ Length of the diameter $AB = \sqrt{a^2+0+a^2} = a\sqrt{2}$

Radius of the circle $r = \frac{a}{\sqrt{2}}$.



W.k.t stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\bar{F}\cdot\bar{n}\,ds$.

$$\text{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

We have $\bar{F} = F_1 i + F_2 j + F_3 k$

i.e $\bar{F} = yi + zj + xk$

$$\text{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$\text{curl}\,\bar{F} = i\left(\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(z)\right) - j\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(y)\right) + k\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y)\right)$$

$$\text{curl}\,F = -i - j - k.$$

Let $\bar{n}$ be the unit normal to the surface ϕ. $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Let $\phi = x + z - a$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i + k, \quad |\nabla\phi| = \sqrt{2}.$$

$$\bar{n} = \frac{i+k}{\sqrt{2}}$$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds$$

$$= -\int_S (i+j+k)\cdot\frac{i+k}{\sqrt{2}}\,ds$$

$$= -\frac{1}{\sqrt{2}}\int_S (1+1)\,ds$$

$$= -\sqrt{2}\int_S ds$$

$= -\sqrt{2}$. Area of the circle

$$= -\sqrt{2}\cdot\frac{\pi a^2}{2}$$

$$= \frac{\pi a^2}{\sqrt{2}}.$$

→ Use stoke's theorem to evaluate $\int_S \operatorname{curl}\bar{F}.\bar{n}\,ds$ where $\bar{F}=2yi+(x-2zx)j+xyk$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the XY-plane.

Sol: Given that $\bar{F}=2yi+(x-2zx)j+xyk$

We have $\bar{F}=F_1 i+F_2 j\, F_3 k$. $F_1=2yi$ $F_2=x-2xz$ $F_3=xy$.

Given that S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the XY-plane.

WKT stokes theorem $\oint_C \bar{F}.d\bar{r}=\int_S \operatorname{curl}\bar{F}.\bar{n}\,ds$.

The boundary C of the surface S is the circle $x^2+y^2=a^2$, $z=0$

$[\because$ In XY-plane $z=0]$

The parametric equations of C are $x=a\cos\theta$ $y=a\sin\theta$, $z=0$

$dx=-a\sin\theta\, d\theta$ $dy=a\cos\theta\, d\theta$

$\bar{F}=2yi+xj+xyk$ $[\because z=0$ in XY-plane$]$

$\bar{F}=2a\sin\theta\, i+a\cos\theta\, j+a^2\sin\theta\cos\theta\, k$.

We have $\bar{r}=xi+yj$ $[\because$ In XY-plane $z=0]$

$d\bar{r}=dx\, i+dy\, j$

$d\bar{r}=[-a\sin\theta i+a\cos\theta j]d\theta$

$\bar{F}.d\bar{r}=[2a\sin\theta\, i+a\cos\theta\, j+a^2\sin\theta\cos\theta\, k].[-a\sin\theta\, d\theta\, i+a\cos\theta\, d\theta\, j]$

$\bar{F}.d\bar{r}=-2a^2\sin^2\theta\, d\theta+a^2\cos^2\theta\, d\theta$.

$\bar{F}.d\bar{r}=\left[-a^2(1-\cos 2\theta)+\frac{a^2}{2}(1+\cos 2\theta)\right]d\theta$

θ varies from 0 to 2π.

$\therefore$ θ limits $\theta=0$, $\theta=2\pi$.

$$\oint_C \bar{F}.d\bar{r}=\int_{\theta=0}^{\theta=2\pi}\left[-a^2(1-\cos 2\theta)+\frac{a^2}{2}(1+\cos 2\theta)\right]d\theta$$

$$=\left[-a^2\left(\theta-\frac{\sin 2\theta}{2}\right)+\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\right]_{\theta=0}^{\theta=2\pi}$$

$$=-\pi a^2.$$

→ Use stokes theorem evaluate $\int_c [(x+y)dx + (2x-z)dy + (y+z)dz]$ where c is the boundary of the triangle with vertices $(2,0,0)$ $(0,3,0)$ and $(0,0,6)$.

Sol.- Given that $\int_c (x+y)dx + (2x-z)dy + (y+z)dz$

Let $\bar{F}.d\bar{r} = (x+y)dx + (2x-z)dy + (y+z)dz$

$\bar{F}.d\bar{r} = [(x+y)i + (2x-z)j + (y+z)k].[i\,dx + j\,dy + k\,dz]$

$\bar{F} = (x+y)i + (2x-z)j + (y+z)k$.

W.k.t stokes theorem. $\oint_c \bar{F}.d\bar{r} = \int_S \text{curl}\,\bar{F}.\bar{n}\,ds$.

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\bar{F} = F_1 i + F_2 j + F_3 k$

$F_1 = x+y$

$F_2 = 2x-z$

$F_3 = y+z$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right] - j\left[\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y)\right] + k\left[\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y)\right]$$

$\text{curl}\,\bar{F} = 2i + k$

An equation of the plane through $A(2,0,0)$ $B(0,3,0)$ $C(0,0,6)$ is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \quad (\text{or}) \quad 3x + 2y + z = 6.$$

Normal to the plane is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Let $\phi = 3x + 2y + z - 6$

$\nabla\phi = 3i + 2j + k$.

Unit normal vector $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3i+2j+k}{\sqrt{9+4+1}} = \frac{3i+2j+k}{\sqrt{14}}$.

$$\int_c \bar{F}.d\bar{r} = \int_c (x+y)dx + (2x-z)dy + (y+z)dz$$

$$= \int_S \text{curl}\,\bar{F}.\bar{n}\,ds$$

$$= \int_S (2i+k)\cdot\frac{(3i+2j+k)}{\sqrt{14}}\,ds$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{1}{\sqrt{14}}(6+1)\int_S ds$$

$$= \frac{7}{\sqrt{14}}\int_S ds$$

$$= \frac{7}{\sqrt{14}}\int_S ds = \frac{7}{\sqrt{14}}(\text{Area of } \Delta ABC)$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{7}{\sqrt{14}}|AB\times AC|\cdot\frac{1}{2} \quad \text{—— (1)}$$

We have $A(2,0,0)$ $B(0,3,0)$ $C(0,0,6)$

$AB = (-2,3,0)$ $AC = (-2,0,6)$

$\bar{AB} = -2i+3j$ $\bar{AC} = -2i+6k.$

$$AB\times AC = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix}$$

$$AB\times AC = i(18) - j(-12) + k(6)$$

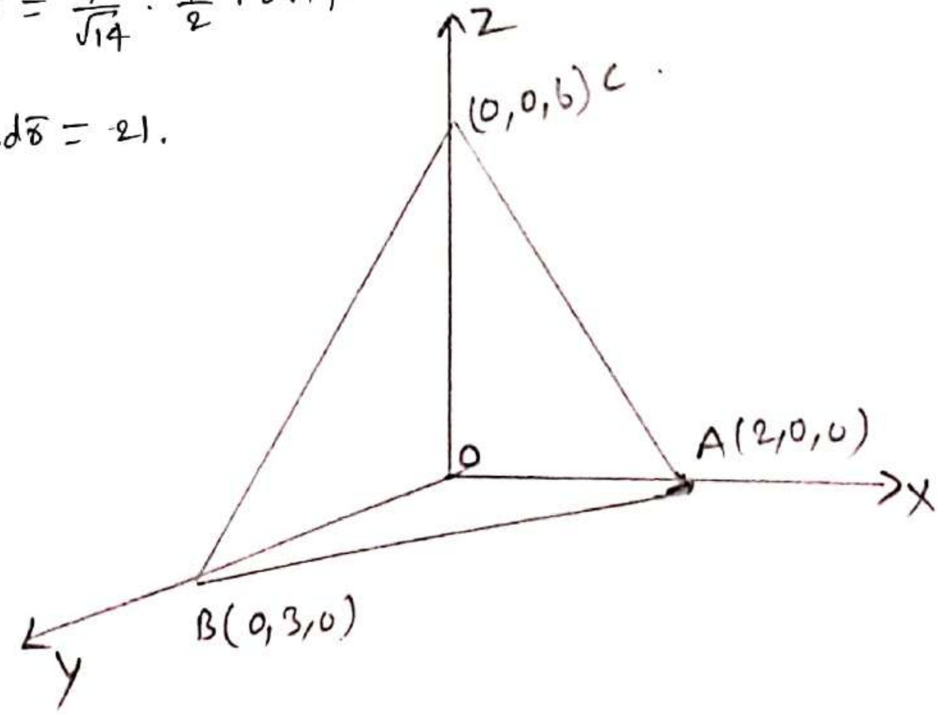
$$AB\times AC = 6(3i+2j+k)$$

$$|AB\times AC| = 6\sqrt{9+4+1} = 6\sqrt{14}. \quad \text{—— (2)}$$

From (1) and (2)

$$\int_C \bar{F}\cdot d\bar{r} = \frac{7}{\sqrt{14}}\cdot\frac{1}{2}\cdot 6\sqrt{14}$$

$$\int_C \bar{F}\cdot d\bar{r} = 21.$$



Volume Integrals :-

Consider a closed surface in space enclosing a volume V. Then, integrals of the form $\iiint_V \bar{F}\,dv$ and $\iiint_V \phi\,dv$ [$\bar{F}$ is a vector function, ϕ is a scalar function] are examples of volume integrals.

Expression of volume integral as the limit of a sum :-

Let $\bar{F}$ be a continuous vector function. Let S be a surface enclosing the region D. Divide this region D into a finite number of subregions $D_1, D_2, D_3 \ldots D_n$.

Let ΔV_i be the volume of the subregion D_i enclosing any point whose position vector is $\bar{r}_i$.

Consider the sum $V = \sum_{i=1}^{n} \bar{F}(\bar{r}_i)\,\Delta V_i$

The limit of this sum as $n \to \infty$ such that $\Delta V_i \to 0$ is called the volume integral of $\bar{F}$ over D and is denoted by $\iiint_D \bar{F}\,dv$.

If $\bar{F} = F_1(x,y,z)\,i + F_2(x,y,z)\,j + F_3(x,y,z)\,k$. So that $dv = dx\,dy\,dz$

$$\iiint_D \bar{F}\,dv = i\iiint_D F_1(x,y,z)\,dx\,dy\,dz + j\iiint_D F_2(x,y,z)\,dx\,dy\,dz + k\iiint_D F_3(x,y,z)\,dx\,dy\,dz$$

→ Evaluate $\iiint_V \phi \, dv$ taken over the rectangular parallelopiped $0 \le x < a$, $0 \le y < b$, $0 \le z < c$ and $\phi = 2(x+y+z)$.

Sol:- Given that $\phi = 2(x+y+z)$

Given that rectangular parallelopiped $0 \le x < a$, $0 \le y < b$, $0 \le z < c$.

$$I = \iiint \phi \, dv. \qquad [\because dv = dz\,dy\,dx]$$

$$I = \int_{x=0}^{x=a} \int_{y=0}^{y=b} \int_{z=0}^{z=c} [2(x+y+z)\, dz]\, dy\, dx.$$

$$= \int_{x=0}^{x=a} \int_{y=0}^{y=b} 2\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{z=c} dy\, dx.$$

$$= \int_{x=0}^{x=a} \int_{y=0}^{y=b} \{[2cx + 2cy + c^2]\, dy\}\, dx$$

$$= \int_{x=0}^{x=a} [2cxy + cy^2 + c^2y]_{y=0}^{y=b}\, dx.$$

$$= \int_{x=0}^{x=a} [2bcx + cb^2 + c^2b]\, dx$$

$$= \left[2bc\frac{x^2}{2} + cb^2x + c^2bx\right]_{x=0}^{x=a}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c)$$

→ If $\bar{F} = (2x^2-3z)i - 2xy\,j - 4x\,k$ evaluate $\iiint_V \nabla\cdot\bar{F}\,dv$ where V is the closed region bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

Sol:- Given that $\bar{F} = (2x^2-3z)i - 2xy\,j - 4x\,k$, $\bar{F} = F_1 i + F_2 j + F_3 k$.

We have to find $\int_V \nabla\cdot\bar{F}\,dv$

$$\nabla\cdot\bar{F} = \operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$F_1 = 2x^2-3z \quad F_2 = -2xy \quad F_3 = -4x.$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(2x^2-3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$\nabla\cdot\bar{F} = 4x-2x = 2x$$

Given that the region bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

We have $2x+2y+z=4$ —① $\Rightarrow z = 4-2x-2y$.

∴ z limits $z=0$, $z=4-2x-2y$.

Given $2x+2y+z=4$

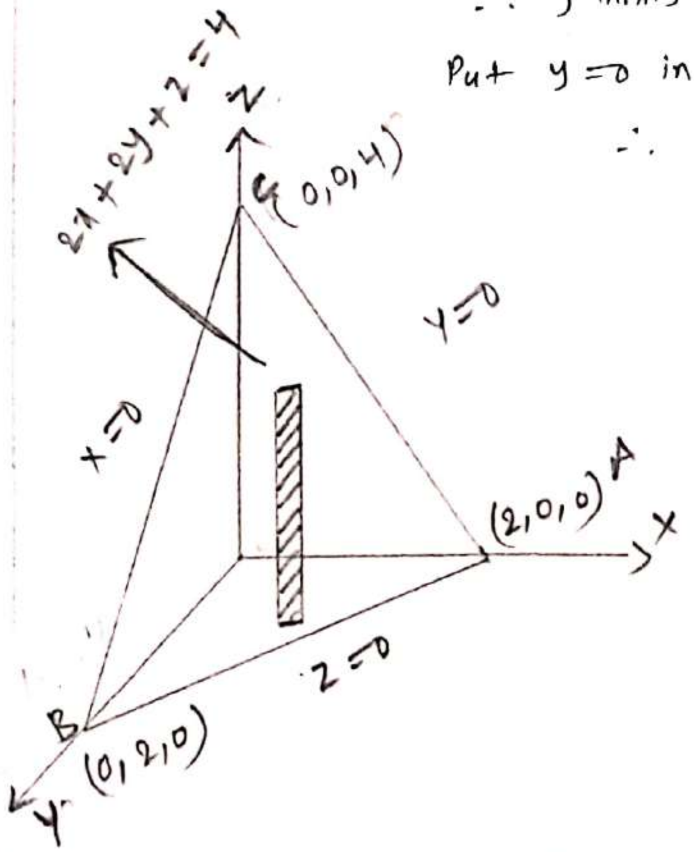
Put $z=0$, $x+y=2$ —②

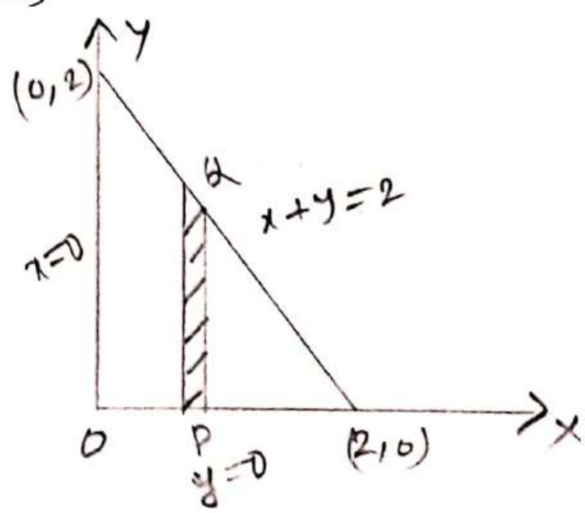
$y = 2-x$

∴ y limits $y=0$ $y=2-x$.

Put $y=0$ in ②, we get $x=2$

∴ x limits $x=0$ $x=2$.





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$$\iiint_V \nabla\cdot F\,dv = \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\int_{z=0}^{z=4-2x-2y} 2x\,dz\,dy\,dx.$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\left[\int_{z=0}^{z=4-2x-2y} 2x\,dz\right]dy\,dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\left[2xz\right]_{z=0}^{z=4-2x-2y} dy\,dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} 2x(4-2x-2y)\,dy\,dx$$

$$= \int_{x=0}^{x=2}\left[\int_{y=0}^{y=2-x}\left[8x-4x^2-4xy\right]dy\right]dx$$

$$= \int_{x=0}^{x=2}\left[8xy-4x^2y-2xy^2\right]_{y=0}^{y=2-x} dx.$$

$$= \int_{x=0}^{x=2}\left[8x(2-x)-4x^2(2-x)-2x(2-x)^2\right]dx$$

$$= \int_{x=0}^{x=2}\left[8x-8x^2+2x^3\right]dx$$

$$= \left[4x^2-\frac{8x^3}{3}+\frac{2x^4}{4}\right]_{x=0}^{x=2}$$

$$= 16-\frac{64}{3}+8$$

$$\iiint_V \nabla\cdot F\,dv = \frac{8}{3}$$

→ If $\bar{F} = (2x^2-3z)i - 2xy\,j + 4x\,k$ then evaluate $\int_V \nabla\times\bar{F}\,dv$ where V is the closed region bounded by $x=0,\ y=0,\ z=0$ $2x+2y+z=4$.

Sol:- Given that $\bar{F} = (2x^2-3z)i - 2xy\,j - 4x\,k$. $\bar{F} = F_1 i + F_2 j + F_3 k$.

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right] - j\left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2-3z)\right] + k\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2-3z)\right]$$

$$\text{curl}\,\bar{F} = j - 2y\,k.$$

Given that the region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$.

We have $2x+2y+z=4$ — ① $\Rightarrow z = 4-2x-2y$

$\therefore$ z limits $z=0,\ z=4-2x-2y$

Given $2x+2y+z=4$

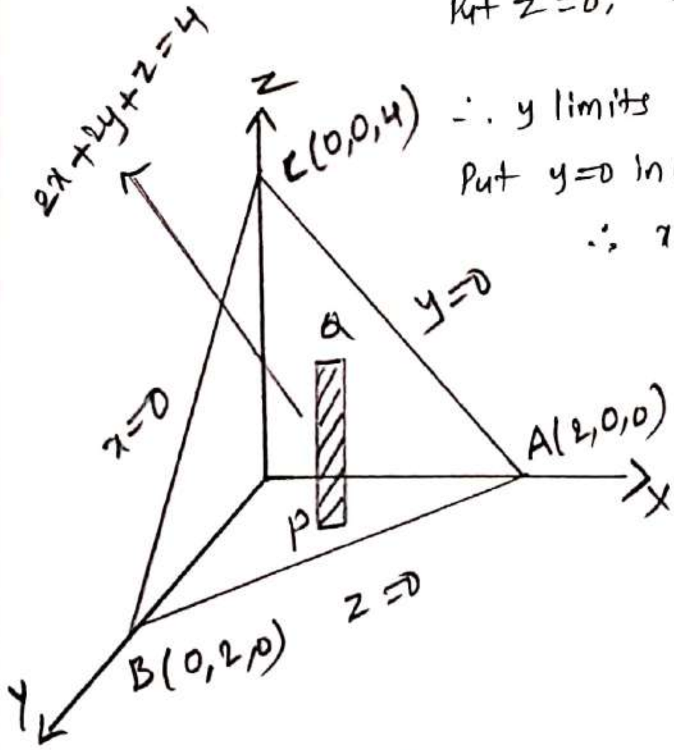
Put $z=0$, $x+y=2$ — ②

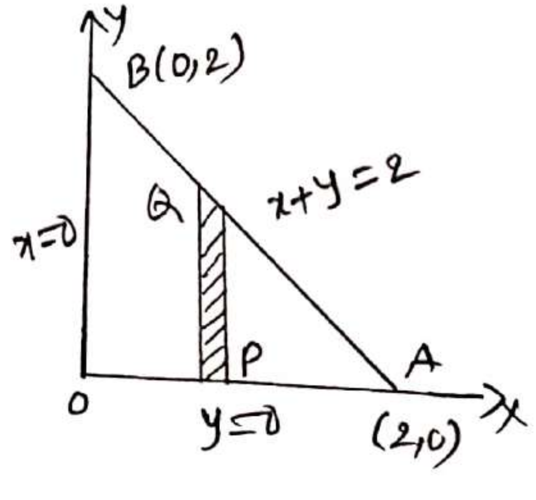
$\Rightarrow y = 2-x$.

$\therefore$ y limits $y=0,\ y=2-x$

Put $y=0$ in ②, we get $x=2$

$\therefore$ x limits $x=0,\ x=2$





$$\int_V \nabla\times F\,dv = \iiint_V (j-2yk)\,dx\,dy\,dz$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\int_{z=0}^{z=4-2x-2y} (j-2yk)\,dz\,dy\,dx.$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} (j-2yk)\,[z]_{z=0}^{z=4-2x-2y}\,dy\,dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} (j-2yk)(4-2x-2y)\,dy\,dx.$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} \left\{ j[(4-2x)-2y] - k[(4-2x)2y-4y^2] \right\} dy\,dx$$

$$= \int_{x=0}^{x=2} j\left[(4-2x)y - y^2\right]_{y=0}^{y=2-x} - k\int_{x=0}^{x=2}\left[(4-2x)y^2 - 4\frac{y^3}{3}\right]_{y=0}^{y=2-x} dx.$$

$$= j\int_{x=0}^{x=2} (2-x)^2\,dx - k\int_{x=0}^{x=2} \frac{2}{3}(2-x)^3\,dx.$$

$$= j\left[\frac{(2-x)^3}{-3}\right]_{x=0}^{x=2} - k\frac{2}{3}\left[\frac{(2-x)^4}{-4}\right]_{x=0}^{x=2}$$

$$\int_V (\nabla\times F)\,dv = \frac{8}{3}(j-k)$$

Gauss's Divergence Theorem :-

(Transformation between surface Integral and Volume Integral)

Let s be a closed surface enclosing a volume v. If $\bar{F}$ is a continuo-
-usly differentiable vector point function, then

$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$ where n is the outward drawn normal vector
at any point of s.

→ Verify the divergence theorem for $\bar{F} = (4xy)i - y^2 j + (xz)k$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0$ and $z=1$.

Sol:- Given that $\bar{F} = (4xy)i - y^2 j + (xz)k$

Given that the cube bounded by $x=0, x=1, y=0, y=1, z=0$ and $z=1$.

W.k.t Gauss Divergence Theorem.

$$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds .$$

To evaluate $\int_V \operatorname{div}\bar{F}\, dv$:-

We have $\bar{F} = (4xy)i - y^2 j + (xz)k$ $\quad \bar{F} = F_1 i + F_2 j + F_3 k.$

$F_1 = 4xy \quad F_2 = -y^2 \quad F_3 = xz$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(xz)$$

$$\operatorname{div}\bar{F} = 4y - 2y + x$$

$$\operatorname{div}F = x + 2y .$$

$$\int_V \operatorname{div}\bar{F} = \int_V (x+2y)\, dv \qquad \because dv = dx\,dy\,dz$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1}\left[\int_{z=0}^{z=1}(x+2y)\,dz\right]dy\,dx .$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1}(x+2y)\left[z\right]_{z=0}^{z=1} dy\,dx$$

$$= \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1}(x+2y)\,dy\right]dx.$$

$$= \int_{x=0}^{x=1}\left[xy+y^2\right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=1}(x+1)\,dx$$

$$= \left[\frac{x^2}{2}+x\right]_{x=0}^{x=1}$$

$$= \frac{1}{2}+1$$

$$\int \operatorname{div}\bar{F}\,dv = \frac{3}{2}.$$

<u>To evaluate $\int_s \bar{F}.\bar{n}\,ds$</u> :-

To evaluate the surface integral divide the closed surface s of the cube into 6 parts.

i.e S_1 : The face ABFG

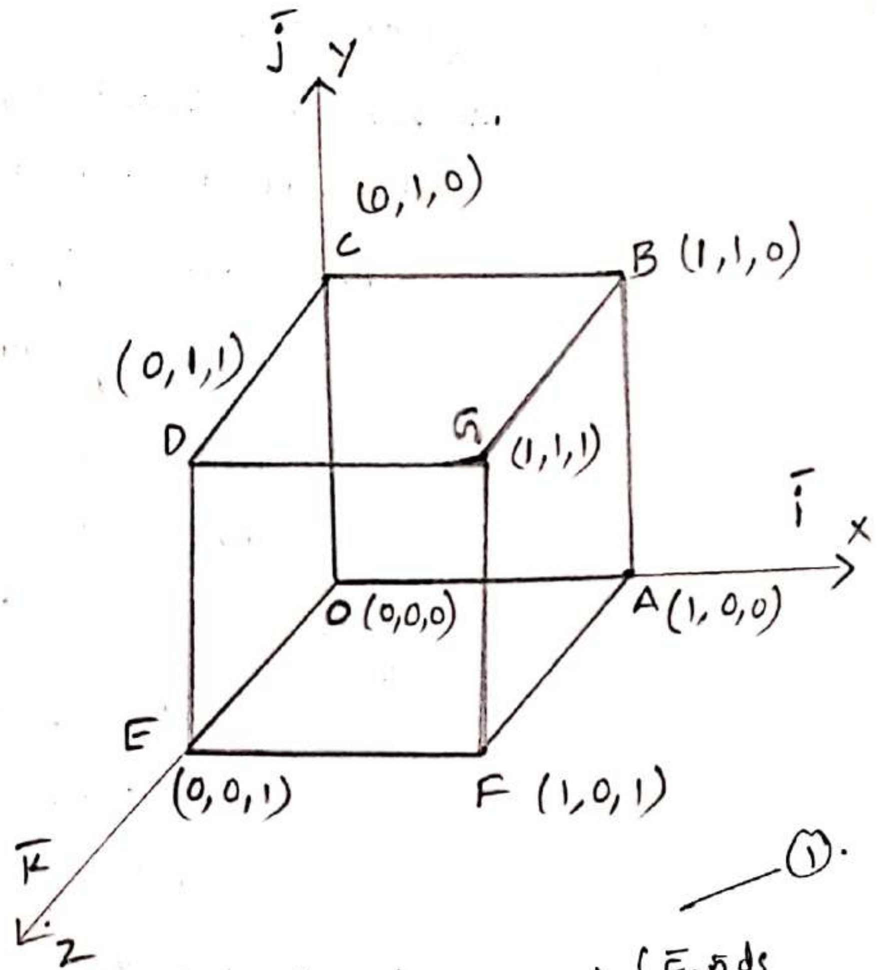
S_2 : The face OCDE

S_3 : The face BCDG.

S_4 : The face OAEF.

S_5 : The face DEFG

S_6 : The face OABC.



(1).

$$\therefore \int_s \bar{F}.\bar{n}\,ds = \int_{S_1}\bar{F}.\bar{n}\,ds + \int_{S_2}\bar{F}.\bar{n}\,ds + \int_{S_3}\bar{F}.\bar{n}\,ds + \cdots + \int_{S_6}\bar{F}.\bar{n}\,ds$$

Case (i) To evaluate $\int_{S_1} \bar{F}.\bar{n}\, ds$. (OR) on the face ABFG .

Unit normal vector to S_1 (face ABFG) is $\bar{n} = \bar{i}$

On S_1, $x = 1$,

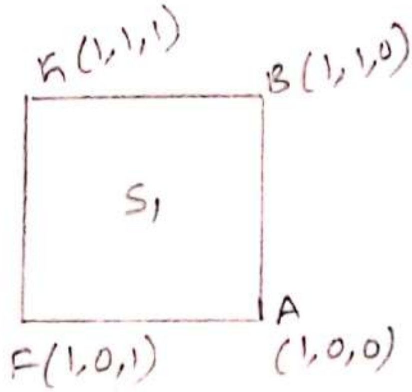
$ds = dy\,dz$. [$\because$ Face ABFG is ||el to YZ plane]

y varies from 0 to 1 .

$\therefore$ y limits $y = 0$, $y = 1$.

z varies from 0 to 1

$\therefore$ z limits $z = 0$, $z = 1$



$\bar{F}.\bar{n} = [4xy\, i - y^2 j + (xz)k] . i$

$\bar{F}.\bar{n} = 4xy$

$\bar{F}.\bar{n} = 4y$ [$\because x = 1$]

$$\int_{S_1} \bar{F}.\bar{n}\, ds = \int_{S_1} 4y\, dy\, dz = \int_{y=0}^{y=1}\int_{z=0}^{z=1} 4y\, dy\, dz$$

$$= 4\int_{y=0}^{y=1} y\,dy \int_{z=0}^{z=1} dz = 4\left[\frac{y^2}{2}\right]_{y=0}^{y=1} [z]_{z=0}^{z=1}$$

$$= 4\left(\frac{1}{2} - 0\right)(1-0)$$

$$\int_{S_1} \bar{F}.\bar{n}\, ds = 2 \quad \text{—— (2)}$$

Case (ii) :- To evaluate $\int_{S_2} \bar{F}.\bar{n}\, ds$ (OR) on the face OCDE :

Unit normal vector to (S_2 face OCDE) is $\bar{n} = -i$

On S_2, $x = 0$

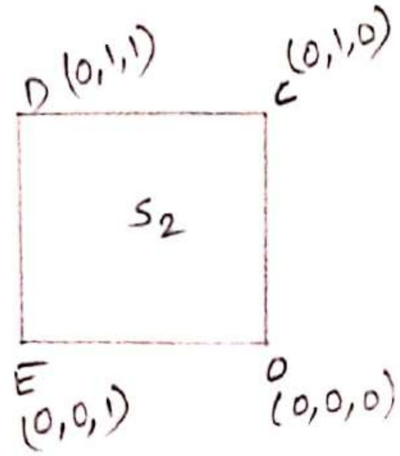
$ds = dy\,dz$ [$\because$ Face OCDE is in YZ plane]

y varies from 0 to 1 .

$\therefore$ y limits $y = 0$, $y = 1$

z varies from 0 to 1 .

$\therefore$ z limits $z = 0$, $z = 1$



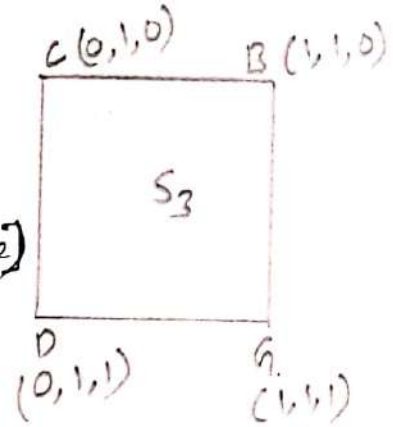
$\bar{F}.\bar{n} = [4xy\, i - y^2 j + (xz)k] . (-i)$

$\bar{F}.\bar{n} = -4xy$

$\bar{F}.\bar{n} = 0 \quad [\because x = 0]$

$$\int_{S_2} \bar{F}.\bar{n}\, ds = 0 \quad \text{—— (3)}$$

<u>Case (iii)</u> :- To evaluate $\int_{S_3} \bar{F}.\bar{n}\, ds$ (OR) On the face BCDG.



Unit normal vector to S_3 (face BCDG) is $\bar{n} = j$

On S_3, $y = 1$

$ds = dx\,dz$ [$\because$ Face BGDG is ||lel to xz plane]

x varies from 0 to 1.

$\therefore$ x limits $x = 0$, $x = 1$

z varies from 0 to 1.

$\therefore$ z limits $z = 0$, $z = 1$.

$$\bar{F}.\bar{n} = [4xy\, i - y^2 j + (xz) k] . j$$

$$\bar{F}.\bar{n} = -y^2$$

$$\bar{F}.\bar{n} = -1 \quad [\because y = 1]$$

$$\int_{S_3} \bar{F}.\bar{n}\, ds = \int_{S_3} -1 . dx\, dz = \int_{x=0}^{x=1} \int_{z=0}^{z=1} -1 . dx\,dz$$

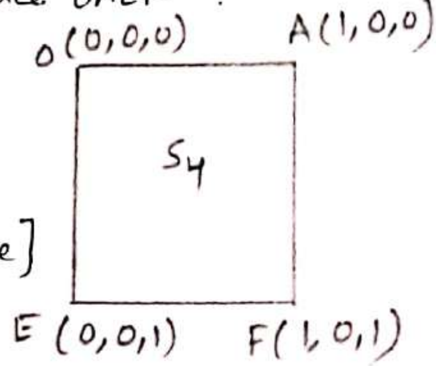
$$= - \int_{x=0}^{x=1} dx \int_{z=0}^{z=1} dz$$

$$= - [x]_{x=0}^{x=1} [z]_{z=0}^{z=1}$$

$$= -(1-0)(1-0)$$

$$\int_{S_3} \bar{F}.\bar{n}\, ds = -1 \quad \text{—— (4)}$$

<u>Case (iv)</u> :- To evaluate $\int_{S_4} \bar{F}.\bar{n}\, ds$ (OR) On the face OAEF :-



Unit normal vector to S_4 (face OAEF) is $\bar{n} = -j$

On S_4, $y = 0$.

$ds = dx\,dz$ [$\because$ Face OAEF is in xz plane]

x varies from 0 to 1

$\therefore$ x limits $x=0,\ x=1$.

z varies from 0 to 1.

$\therefore$ z limits $z=0,\ z=1$.

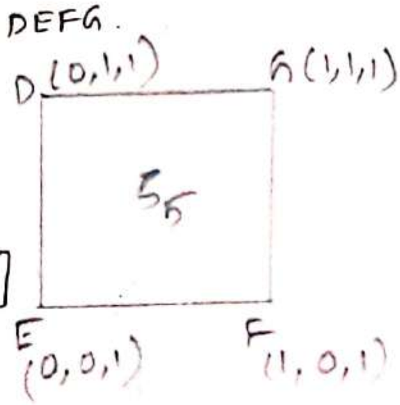
$$\bar{F}\cdot\bar{n} = [4xy\,i - y^2 j + (xz)k]\cdot(-j)$$

$$\bar{F}\cdot\bar{n} = y^2$$

$$\bar{F}\cdot\bar{n} = 0 \quad [\because y=0]$$

$$\therefore \int_{S_4} \bar{F}\cdot\bar{n}\,ds = 0 \text{ ——— } (5)$$

Case (v) :- To evaluate $\int_{S_5} \bar{F}\cdot\bar{n}\,ds$ (OR) on the face DEFG.



Unit normal vector to S_4 (face DEFG) is $\bar{n}=\bar{k}$

on S_5, $z=1$

$ds = dx\,dy$ [$\because$ Face DEFG is ||el to xy-plane]

x varies from 0 to 1

$\therefore$ x limits $x=0,\ x=1$

y varies from 0 to 1

$\therefore$ y limits $y=0\ \ y=1$

$$\bar{F}\cdot\bar{n} = [4xy\,i - y^2 j + (xz)k]\cdot k$$

$$\bar{F}\cdot\bar{n} = xz$$

$$\bar{F}\cdot\bar{n} = x \quad [\because z=1]$$

$$\int_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_{S_5} x\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1} x\,dx\,dy$$

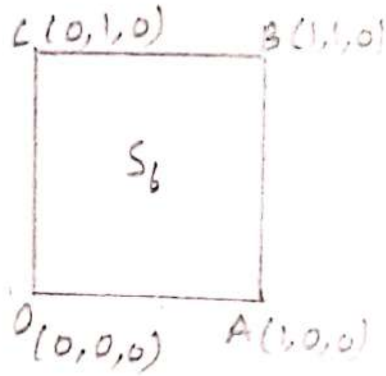
$$= \int_{x=0}^{x=1} x\,dx \int_{y=0}^{y=1} dy$$

$$= \left[\frac{x^2}{2}\right]_{x=0}^{x=1} [y]_{y=0}^{y=1}$$

$$= \left(\frac{1}{2}-0\right)(1-0)$$

$$\int_{S_5} \bar{F}\cdot\bar{n}\,ds = \frac{1}{2} \text{ ——— } (6)$$

Case (vi) To evaluate $\int_{S_6} \bar{F}\cdot\bar{n}\,ds$ (OR) on the face OABC.



Unit normal vector to S_6 (face OABC) is $\bar{n} = -\bar{k}$

On S_6, $z = 0$

$ds = dx\,dy$ [$\because$ Face OABC is in xy-plane]

x varies from 0 to 1

$\therefore$ x limits $x=0,\ x=1$

y varies from 0 to 1

$\therefore$ y limits $y=0,\ y=1$

$\bar{F}\cdot\bar{n} = [4xy\,i - y^2 j + (xz)k]\cdot(-k)$

$\bar{F}\cdot\bar{n} = -xz$

$\bar{F}\cdot\bar{n} = 0$ [$\because z=0$]

$\int_{S_6} \bar{F}\cdot\bar{n}\,ds = 0$ —— (7)

Sub. (2) (3) (4) (5). (6) and (7) in (1), we get

$\int_S \bar{F}\cdot\bar{n}\,ds = 2+0-1+0+\frac{1}{2}+0$

$\int_S \bar{F}\cdot\bar{n}\,ds = \frac{3}{2}$.

$\therefore \int_V \mathrm{div}\,\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds$

$\therefore$ Gauss Divergence Theorem verified.

Verify Gauss divergence theorem for $f = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$ taken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

Sol:- Given that $f = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$

and the given the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

We know that the Gauss Divergence theorem.

$$\int_S \bar{F}.n\, ds = \iiint_V \nabla.\bar{F}\, dv.$$

To evaluate $\iiint_V \nabla.\bar{F}\, dv$:-

$$\nabla.\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

$$\nabla.\bar{f} = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$\nabla.\bar{f} = 2x + 2y + 2z.$$

$$\iiint_V \nabla.\bar{f}\, dv = \int_{x=0}^{x=a}\int_{y=0}^{y=b}\int_{z=0}^{z=c} 2(x+y+z)\, dz\, dy\, dx.$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=b}\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{z=c} dy\, dx$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=b} cx + cy + \frac{c^2}{2}\Big] dy\, dx.$$

$$= 2\int_{x=0}^{x=a}\left[cxy + c\frac{y^2}{2} + \frac{c^2}{2}y\right]_{y=0}^{y=b} dx$$

$$= 2\int_{x=0}^{x=a}\left[bcx + \frac{cb^2}{2} + \frac{bc^2}{2}\right] dx$$

$$= 2\left[bc\frac{x^2}{2} + \frac{cb^2}{2}x + \frac{bc^2}{2}x\right]_{x=0}^{x=a}$$

$$= 2\left[\frac{bca^2}{2} + \frac{abc^2}{2} + \frac{acb^2}{2}\right]$$

$$\iiint_V (\nabla\cdot f)\, dv = abc(a+b+c) \quad \text{--- (1)}$$

To evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$:-

S is the surface of the rectangular parallelopiped given by

$0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

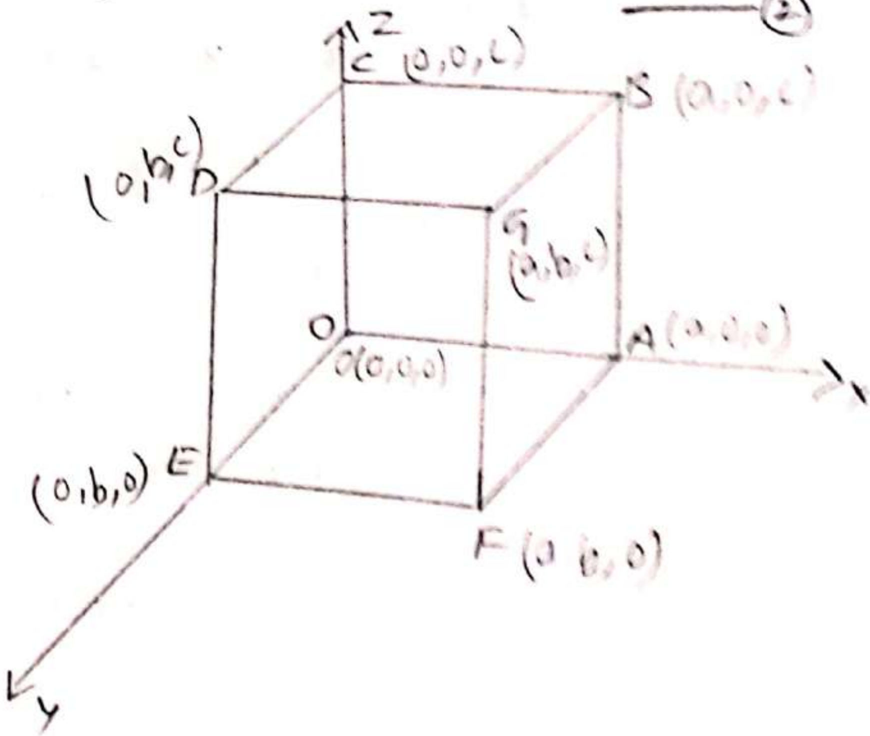
We note that the boundary surface S of the given rectangular parallelopiped is made up of the following six faces.

S_1 : OABC (xz plane) $\qquad$ S_2 : DEGF (opposite to xz plane)

S_3 : OCDE (yz plane) $\qquad$ S_4 : ABFG (opposite to yz plane)

S_5 : OAEF (xy plane) $\qquad$ S_6 : BCDG (opposite to xy plane)

$$\int_S \bar{F}\cdot\bar{n}\, ds = \int_{S_1} \bar{F}\cdot\bar{n}\, ds + \int_{S_2} \bar{F}\cdot\bar{n}\, ds + \int_{S_3} \bar{F}\cdot\bar{n}\, ds + \cdots + \int_{S_6} \bar{F}\cdot\bar{n}\, ds. \quad \text{--- (2)}$$



(i) To evaluate $\int_{S_1} \bar{F}.\bar{n}\, ds$:-

On the face OABC we have $y=0$, $0 \le x \le a$ and $0 \le z \le c$.

The unit outward drawn normal to the face OABC is $\bar{n} = -j$

$$\int_{S_1} \bar{F}.\bar{n}\, ds = \int_{x=0}^{x=a}\int_{z=0}^{z=c} \left[(x^2-0z)i + (0.zx)j + (z^2-x.0)k\right].(-j)\, dz\, dx.$$

$$= \int_{x=0}^{x=a}\int_{z=0}^{z=c} zx\, dz\, dx$$

$$= \int_{x=0}^{x=a} x\, dx \left[\frac{z^2}{2}\right]_{z=0}^{z=c}$$

$$= \frac{c^2}{2}\int_{x=0}^{x=a} x\, dx = \frac{c^2}{2}\left[\frac{x^2}{2}\right]_{x=0}^{x=a}$$

$$= \frac{a^2c^2}{4}. \quad \text{——— (3)}$$

(ii) To evaluate $\int_{S_2} \bar{F}.\bar{n}\, ds$:—

On the face DEFG we have $y=b$, $0 \le x \le a$, $0 \le z \le c$.

The unit oward drawn normal to the face DEFG is $\bar{n} = j$

$$\int_{S_2} \bar{F}.\bar{n}\, ds = \int_{x=0}^{x=a}\int_{z=0}^{z=c} \left[(x^2-bz)i + (b^2-zx)j + (z^2-bx)k\right].\, j\, dz\, dx.$$

$$= \int_{x=0}^{x=a}\int_{z=0}^{z=c} (b^2-zx)\, dz.dx.$$

$$= \int_{x=0}^{x=a}\left[b^2z - \frac{z^2x}{2}\right]_{z=0}^{z=c} dx = \int_{x=0}^{x=a}\left[cb^2 - \frac{c^2x}{2}\right] dx$$

$$= \left[cb^2x - \frac{c^2x^2}{4}\right]_{x=0}^{x=a}$$

$$= ab^2c - \frac{a^2c^2}{4} \quad \text{——— (4)}$$

(iii) To evaluate $\int_{S_3} \bar{F}\cdot\bar{n}\, ds$:—

On the face OCDE we have $x=0$, $0\le y\le b$, $0\le z\le c$.

The unit outward drawn normal to the face OCDE is $\bar{n}=-i$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = \int_{y=0}^{y=b}\int_{z=0}^{z=c} \left[(0-yz)i+(y^2-0.z)j+(z^2-0.y)k\right]\cdot(-i)\,dy\,dz$$

$$= \int_{y=0}^{y=b}\int_{z=0}^{z=c} yz\, dy\, dz$$

$$= \int_{y=0}^{y=b} y\,dy \left[\frac{z^2}{2}\right]_{z=0}^{z=c}$$

$$= \frac{c^2}{2}\int_{y=0}^{y=b} y\,dy = \frac{c^2}{2}\left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = \frac{b^2c^2}{4}. \quad \text{———} (5)$$

(iv) To evaluate $\int_{S_4} \bar{F}\cdot\bar{n}\, ds$:-

On the face ABFG we have $x=a$, $0\le y\le b$, $0\le z\le c$.

The unit outward drawn normal to the face ABFG is $\bar{n}=\bar{i}$

$$\int_{S_4} \bar{F}\cdot\bar{n}\, ds = \int_{y=0}^{y=b}\int_{z=0}^{z=c} \left[(a^2-yz)i+(y^2-az)j+(z^2-ay)k\right]\cdot i\, dy\,dz$$

$$= \int_{y=0}^{y=b}\int_{z=0}^{z=c} (a^2-yz)\,dy\,dz$$

$$= \int_{y=0}^{y=b} \left[a^2z-\frac{yz^2}{2}\right]_{z=0}^{z=c} dy = \int_{y=0}^{y=b}\left[ca^2-\frac{c^2y}{2}\right]dy$$

$$= \left[ca^2y-\frac{c^2y^2}{4}\right]_{y=0}^{y=b}$$

$$= a^2bc-\frac{c^2b^2}{4}. \quad \text{———} (6)$$

(vi) To evaluate $\int_{S_6} \bar{F}\cdot\bar{n}\, ds$:-

On the face BCDG we have $z = c$, $0 \le x \le a$, $0 \le y \le b$.

The unit outward drawn normal to the face BCDG is $\bar{n} = \bar{k}$

$$\int_{S_6} \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{x=a}\int_{y=0}^{y=b} \left[(x^2 - cy)i + (y^2 - cz)j + (c^2 - xy)k\right]\cdot \bar{k}\, dx\, dy$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b} (c^2 - xy)\, dx\, dy.$$

$$= \int_{x=0}^{x=a} \left[c^2 y - \frac{xy^2}{2}\right]_{y=0}^{y=b} dx.$$

$$= \int_{x=0}^{x=a} \left[bc^2 - \frac{b^2 x}{2}\right] dx$$

$$= \left[bc^2 x - \frac{b^2 x^2}{4}\right]_{x=0}^{x=a}$$

$$= abc^2 - \frac{a^2 b^2}{4}. \quad \text{—— (7)}$$

(VI) To evaluate $\int_{S_5} \bar{F}\cdot\bar{n}\, ds$:—

On the face OAEF we have $z = 0$, $0 \le x \le a$, $0 \le y \le b$.

The unit outward drawn normal to the face OAEF is $\bar{n} = -\bar{k}$

$$\int_{S_5} \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{x=a}\int_{y=0}^{y=b} \left[(x^2 - 0.y)i + (y^2 - 0.x)j + (0 - xy)k\right].\, (-\bar{k})\, dx\, dy$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=b} xy\, dx\, dy$$

$$= \int_{x=0}^{x=a} x\,dx \left[\frac{y^2}{2}\right]_{y=0}^{y=b}$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}\right]_{x=0}^{x=a}$$

$$= \frac{a^2b^2}{4} \quad \text{—— (8)}$$

Sub. (3), (4) ... and (8) in (2), we get

$$\int_S \bar{F}\cdot\bar{n}\,ds = \frac{a^2c^2}{4} + a^2bc - \frac{a^2c^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{c^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = abc(a+b+c) \quad \text{—— (9)}$$

$\therefore$ From (1) and (9).

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

$\therefore$ Gauss Divergence theorem verified.

Verify the divergence theorem for $F = 4xy\,i - y^2 j + xz\,k$ over the cube bounded by $x=0$, $x=1$, $y=0$ $y=1$, $z=0$ and $z=1$

Sol: Given that $F = 4xy\,i - y^2 j + xz\,k$.

The cube is bounded by $x=0$ $x=1$ $y=0$ $y=1$ $z=0$ and $z=1$

By Gauss Divergence theorem $\iint_S F.n\,ds = \iiint_V \nabla.F\,dv$.

To Evaluate $\iiint_V \nabla.F\,dv$:-

$$\nabla.F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial(4xy)}{\partial x} + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(xz)$$

$$= 4y - 2y + x = x + 2y$$

$$\nabla.F = x + 2y.$$

$$\iiint_V \nabla.F\,dv = \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} (x+2y)\,dz\,dy\,dx.$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} (x+2y)\,dy\,dx.\left[z\right]_{z=0}^{z=1}$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1} (x+2y)\,dy.\,dx$$

$$= \int_{x=0}^{x=1} \left[xy + y^2\right]_{y=0}^{y=1} dx.$$

$$= \int_{x=0}^{x=1} (x+1)\,dx. = \left[\frac{(x+1)^2}{2}\right]_{x=0}^{x=1}$$

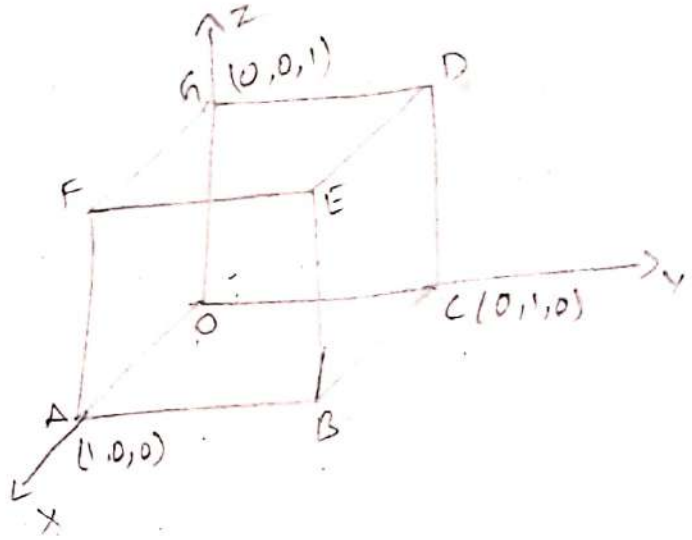
$$= 2 - \frac{1}{2} = \frac{3}{2}.$$

$$\iiint_V \nabla.F\,dv = \frac{3}{2} \quad \text{—} ①$$

To evaluate $\iint_S F.n\, ds$:—

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.

The surface S contains 6 faces.



S_1 : The face BCDE

S_2 : The face OAFG

S_3 : The face ABEF.

S_4 : The face OCDG

S_5 : The face DEFG

S_6 : The face OABC.

The surface integral $\iint_S \bar{F}.\bar{n}\, ds$ is equal to the sum of the surface integrals on the above 6 faces.

$$\therefore \iint_S \bar{F}.\bar{n}\, ds = \iint_{S_1} \bar{F}.\bar{n}\, ds + \iint_{S_2} \bar{F}.\bar{n}\, ds + \cdots + \iint_{S_6} \bar{F}.\bar{n}\, ds. \quad \text{——} (2)$$

The unit outward normals to these faces are $j, -j, i, -i, k$ and $-k$ respectively

(i) on S_1. The face BCDE:

$$\bar{n} = \bar{j} \qquad y = 1. \qquad ds = dx\, dz.$$

$$\bar{F}.\bar{n} = [(4xy)\bar{i} - y^2 j + xz k].j = -y^2 = -1 \quad (\because y=1).$$

$$\iint_{S_1} F.\bar{n}\, ds = \int_{x=0}^{x=1} \int_{z=0}^{z=1} -1\, dx\, dz$$

$$= \int_{x=0}^{x=1} [-z]_0^1\, dx = \int_{x=0}^{x=1} -dx.$$

$$= [-x]_0^1 = -1.$$

$$\therefore \iint_{S_1} \bar{F}.\bar{n}\, ds = -1 \quad \text{——} (3).$$

(ii) on S_2 The face. OAFG.

$$\bar{n} = -j, \quad y=0, \qquad ds = dx\,dz.$$

$$F.\bar{n} = -y^2 = 0.$$

$$\therefore \iint_{S_2} F.\bar{n}\,ds = 0. \quad \text{—} (4)$$

(iii) on S_3 The face ABEF.

$$\bar{n} = \bar{i}, \quad x=1 \qquad ds = dy\,dz.$$

$$F.\bar{n} = [(4xy)i - y^2 j + (z^2)k].i = 4xy$$

$$F.\bar{n} = 4y \qquad (\because x=1)$$

$$\iint_{S_3} F.\bar{n}\,ds = \int_{y=0}^{y=1}\int_{z=0}^{z=1} 4y\,dy\,dz.$$

$$= \int_{y=0}^{y=1} 4y\,dy\,[z]_{z=0}^{z=1}$$

$$= \int_{y=0}^{y=1} 4y\,dy$$

$$= \left[2y^2\right]_{y=0}^{y=1} = 2.$$

$$\iint_{S_3} F.\bar{n}\,ds = 2. \quad \text{—} (5)$$

(iv) on S_4 The face. OCDG.

$$\bar{n} = -i \qquad x=0 \qquad ds = dy\,dz.$$

$$F.\bar{n} = 4xy$$

$$F.\bar{n} = 0 \qquad (\because x=0)$$

$$\therefore \iint_{S_4} F.\bar{n}\,ds = 0. \quad \text{—} (6).$$

On S_5. The face DEFG :-

$$\bar{n} = \bar{k} \qquad z=1 \qquad ds = dx\,dy.$$

$$\bar{F}\cdot\bar{n} = [(4xy)i - y^2 j + (xz)k]\cdot\bar{k} = xz$$

$$\bar{F}\cdot\bar{n} = x \quad (\because z=1)$$

$$\iint_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{x=1}\int_{y=0}^{y=1} x\,dx\,dy$$

$$= \int_{x=0}^{x=1} x\,dx\,[y]_{y=0}^{y=1}$$

$$= \int_{x=0}^{x=1} x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\iint_{S_5} \bar{F}\cdot\bar{n}\,ds = \frac{1}{2}. \quad \text{--- (7)}$$

On S_6 The face OABC.

$$\bar{n} = -\bar{k} \qquad z=0 \qquad ds = dx\,dy.$$

$$\bar{F}\cdot\bar{n} = xz$$

$$\bar{F}\cdot\bar{n} = 0 \quad \because z=0.$$

$$\iint_{S_6} \bar{F}\cdot\bar{n}\,ds = 0. \quad \text{--- (8)}$$

Sub. (3), (4) . . . . (8) in (2), we get

$$\iint_S \bar{F}\cdot\bar{n}\,ds = -1 + 0 + 2 + 0 + \frac{1}{2} + 0. = \frac{3}{2}$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \frac{3}{2} \quad \text{--- (9).}$$

From (1) and (9).

$$\iiint_V \nabla\cdot F\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds.$$

$\therefore$ Gauss Divergence theorem verified.

→ Verify Gauss Divergence Theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.

sol:- Given that $F = x^2 i + y^2 j + z^2 k$

Given that the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.

WKT Gauss Divergence Theorem

$$\int_V \operatorname{div}\bar{F}\, dV = \int_S \bar{F}\cdot\bar{n}\, ds.$$

To evaluate $\int_V \operatorname{div}\bar{F}\, dv$:-

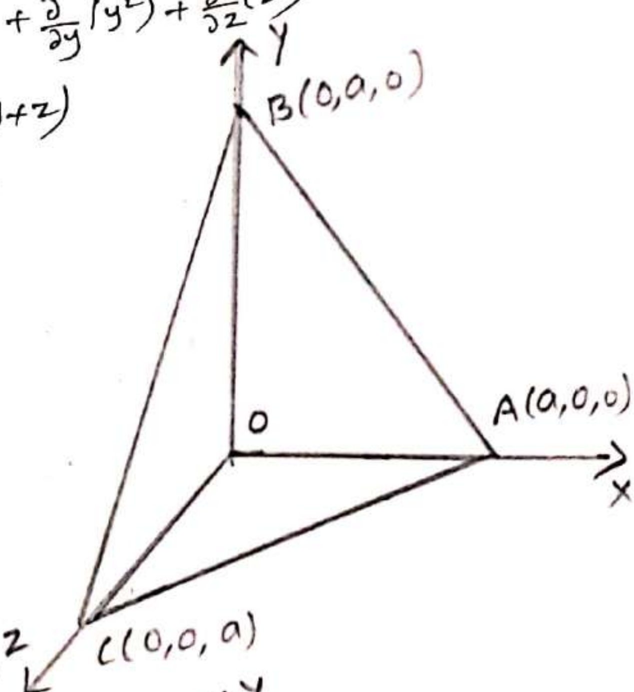
We have $\bar{F} = x^2 i + y^2 j + z^2 k$ $\quad \bar{F} = F_1 i + F_2 j + F_3 k$

$F_1 = x^2 \quad F_2 = y^2 \quad F_3 = z^2$

WKT $\operatorname{div}\bar{F} = \nabla\cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$

$\nabla\cdot\bar{F} = 2(x+y+z)$



We have the plane $x+y+z=a$

$\Rightarrow z = a-x-y$

z limits $z=0,\ z=a-x-y$.

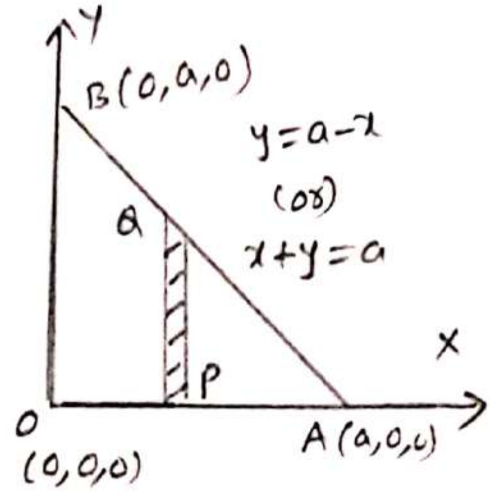
The projection of S in xy-plane is ΔOAB

Here x varies from 0 to a

$\therefore$ x limits $x=0,\ x=a$.

For each x, y varies from a point P on x-axis $(y=0)$ to a point Q on the line $x+y=a$ i.e $y=a-x$

$\therefore$ y limits $y=0,\ y=a-x$



$$\int_V \operatorname{div}\vec{F}\,dv = \int_V 2(x+y+z)\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\int_{z=0}^{z=a-x-y}\left[(x+y)+z\right]dz\,dy\,dx$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\left[z(x+y)+\frac{z^2}{2}\right]_{z=0}^{z=a-x-y} dy\,dx.$$

$$= 2\int_{x=0}^{x=a}\int_{y=0}^{y=a-x}(a-x-y)\left[(x+y)+\frac{a-x-y}{2}\right]dy\,dx.$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x}(a-x-y)(a+x+y)\,dy\,dx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\left[a^2-(x+y)^2\right]dy\,dx$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x}\left[a^2-x^2-y^2-2xy\right]dy\,dx$$

$$= \int_{x=0}^{x=a}\left[a^2y-x^2y-\frac{y^3}{3}-xy^2\right]_{y=0}^{y=a-x}dx$$

$$= \int_{x=0}^{x=a}(a-x)\frac{(2a^2-x^2-ax)}{3}\,dx.$$

$$= \frac{1}{3}\int_{x=0}^{x=a}\left[2a^3-ax^2-a^2x-2a^2x+x^3+ax^2\right]dx$$

$$= \frac{1}{3}\left[2a^3x-a\frac{x^3}{3}-a^2\cdot\frac{x^2}{2}-a^2x^2+\frac{x^4}{4}+\frac{ax^3}{3}\right]_{x=0}^{x=a}$$

$$\int_V \operatorname{div}\vec{F}\,dv = \frac{a^4}{4}.$$

To evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$:-

Let $\phi = x+y+z-a$ be the given plane.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 1 \quad \frac{\partial\phi}{\partial y} = 1 \quad \frac{\partial\phi}{\partial z} = 1$$

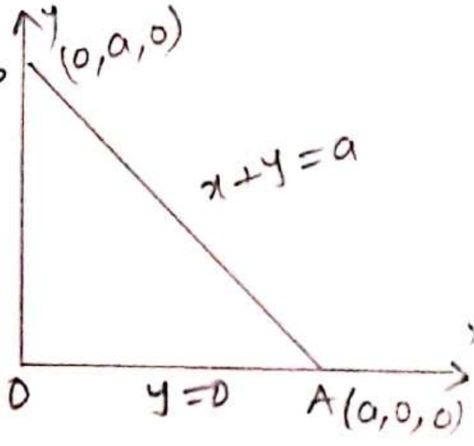
$$\nabla\phi = i+j+k.$$

Unit normal vector $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{i+j+k}{\sqrt{1+1+1}}$

$$\bar{n} = \frac{i+j+k}{\sqrt{3}}$$

$$\bar{F}\cdot\bar{n} = (x^2 i + y^2 j + z^2 k)\cdot\frac{i+j+k}{\sqrt{3}} = \frac{x^2+y^2+z^2}{\sqrt{3}}.$$

Let R be the projection of S in xy-plane. Which is a ΔOAB.



Here x varies from 0 to a.

$\therefore$ x limits $x=0$ $x=a$.

For each x, y varies 0 to $a-x$

$\therefore$ y limits $y=0$ $y=a-x$.

$$\int_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x} \frac{x^2+y^2+z^2}{\sqrt{3}}\,\frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x} \left[x^2+y^2+(a-x-y)^2\right] dy\,dx \quad \left[\because x+y+z=a\right]$$

$$= \int_{x=0}^{x=a}\int_{y=0}^{y=a-x} \left[2x^2+2y^2-2ax+2xy-2ay+a^2\right] dy\,dx$$

$$= \int_{x=0}^{x=a} \left[2x^2y+\frac{2}{3}y^3-2axy+xy^2-ay^2+a^2y\right]_{y=0}^{y=a-x} dx$$

$$= \int_{x=0}^{x=a} \left[2x^2(a-x)+\frac{2}{3}(a-x)^3-2ax(a-x)+x(a-x)^2-a(a-x)^2+a^2(a-x)\right] dx$$

$$= \int_{x=0}^{x=a} \left[-\frac{5}{3} \cdot x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right] dx$$

$$= \left[-\frac{5}{3} \cdot \frac{x^4}{4} + ax^3 - a^2x^2 + \frac{a^3 x}{6}\right]_{x=0}^{x=a}$$

$$= -\frac{5a^4}{12} + a^4 - a^4 + \frac{a^4}{6}$$

$$\int_S \bar{F} \cdot \bar{n}\, ds = \frac{a^4}{4}$$

$$\therefore \int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F} \cdot \bar{n}\, ds.$$

$\therefore$ Gauss Divergence Theorem verified.

→ Use Divergence theorem to evaluate $\iint_S \bar{F}.ds$ where $\bar{F} = x^3 i + y^3 j + z^3 k$ and s is the surface of the sphere $x^2+y^2+z^2 = r^2$.

Sol:- Given that $\bar{F} = x^3 i + y^3 j + z^3 k$, $\bar{F} = F_1 i + F_2 j + F_3 k$.

$$F_1 = x^3 \quad F_2 = y^3 \quad F_3 \, z^3 .$$

We have to find $\iint_S \bar{F}.ds$.

WKT Gauss Divergence theorem

$$\int_S \bar{F}.\bar{n}ds = \int_V div\bar{F}dv .$$

$$div\bar{F} = \nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$div\bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$div\bar{F} = 3(x^2+y^2+z^2)$$

Given that S is the surface of the sphere $x^2+y^2+z^2 = r^2$.

Changing into spherical polar co ordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$
$z = r\cos\theta$, $dxdydz = r^2\sin\theta dr d\theta d\phi$.

r limits $r=0, r=a$ $\quad \theta$ limits $\theta=0, \theta=\pi$ $\quad \phi$ limits $\phi=0, \phi=2\pi$

$$\int_S \bar{F}.\bar{n}ds = \int_V div\bar{F}dv = \iiint_V 3(x^2+y^2+z^2)dxdydz$$

$$= 3\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} r^2(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

$$= 3\int_{r=0}^{r=a} r^4 dr \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta \int_{\phi=0}^{\phi=2\pi} d\phi$$

$$= 3\left[\frac{r^5}{5}\right]_{r=0}^{r=a}\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi}\left[\phi\right]_{\phi=0}^{\phi=2\pi}$$

$$= 3\left[\frac{a^5}{5}-0\right]\left[-\cos\pi+\cos 0\right]\left[2\pi-0\right]$$

$$= \frac{12\pi a^5}{5}$$

→ Apply Divergence theorem to evaluate $\iint_S (x+z)dydz + (y+z)dzdx + (x+y)dxdy$

Where S is the surface of the sphere $x^2+y^2+z^2 = 4$.

sol: Given that $\iint_S (x+z)\,dydz + (y+z)\,dzdx + (x+y)\,dxdy$

WKT $\int_S \bar{F}\cdot\bar{n}\,ds = \iint_S F_1\,dydz + F_2\,dzdx + F_3\,dxdy$.

$F_1 = x+z \qquad F_2 = y+z \qquad F_3 = x+y$

WKT Gauss Divergence Theorem.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv.$$

i.e $\iint_S F_1\,dydz + F_2\,dzdx + F_3\,dxdy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$

$$\frac{\partial F_1}{\partial x} = 1 \quad \frac{\partial F_2}{\partial y} = 1 \quad \frac{\partial F_3}{\partial z} = 0$$

Given that S is the surface of the sphere $x^2+y^2+z^2 = 4$.

$$\iint_S F_1\,dydz + F_2\,dzdx + F_3\,dxdy = \iiint 2\,dxdydz$$

$= 2\int_V dv$

$= 2$ volume of the sphere

$= 2 \cdot \frac{4}{3}\pi r^3$

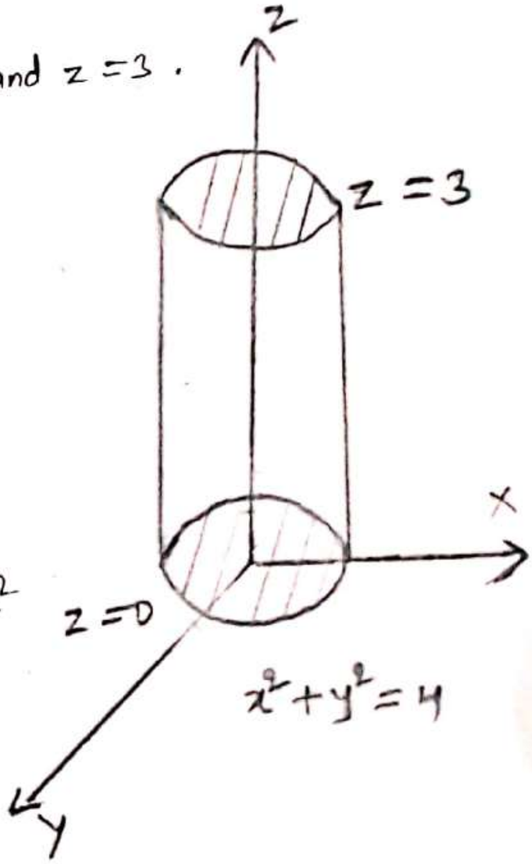
$= 2 \cdot \frac{4}{3}\pi(2)^3$ [∵ Radius of the sphere $r = 2$]

$= \frac{64\pi}{3}$

→ Verify divergence theorem for $\bar{F} = 4xi - 2y^2 j + z^2 k$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Sol:- Given that $\bar{F} = 4xi - 2y^2 j + z^2 k$

The region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.



Wkt Gauss Divergence Theorem

$$\int_V \text{div}\,\bar{F}\,dv = \iint_S \bar{F}\cdot\bar{n}\,ds.$$

To evaluate $\int_V \text{div}\,\bar{F}\,dv$:-

Wkt $\text{div}\,\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

We have $\bar{F} = 4xi - 2y^2 j + z^2 k$

$F_1 = 4x \quad F_2 = -2y^2 \quad F_3 = z^2$

$\frac{\partial F_1}{\partial x} = 4 \quad \frac{\partial F_2}{\partial y} = -4y \quad \frac{\partial F_3}{\partial z} = 2z$

$\text{div}\,\bar{F} = 4 - 4y + 2z.$

We have $z=0$ and $z=3$.

Given that $\underbrace{x^2+y^2=4}_{(1)} \Rightarrow y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

Put $y=0$ in (1), $x^2=4 \Rightarrow x = \pm 2.$

$$\therefore \int_V \text{div}\,\bar{F}\,dv = \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\left[\int_{z=0}^{z=3}(4-4y+2z)\,dz\right]dy\,dx.$$

$$= \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\left[4z-4yz+z^2\right]_{z=0}^{z=3} dy\,dx.$$

$$= \int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\left[(21-12y)\right]dy.\,dx.$$

$$= \int_{x=-2}^{x=2}\left[21y-6y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx.$$

$$= \int_{x=-2}^{x=2} \left[21\sqrt{4-x^2} - 6(4-x^2) - \left(-21\sqrt{4-x^2} - 6(4-x^2)\right)\right] dx$$

$$= \int_{x=-2}^{x=2} 42\sqrt{4-x^2}\, dx .$$

Put $x = 2\cos\theta$

$dx = -2\sin\theta\, d\theta$

when $x=-2$, $\theta=\pi$

when $x=2$, $\theta=0$.

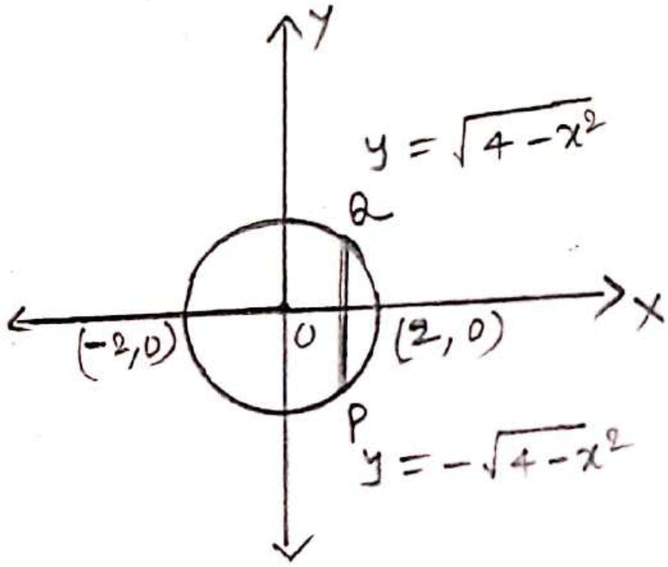
$$= 42\int_{\theta=\pi}^{\theta=0} \sqrt{4-4\cos^2\theta}\,(-2\sin\theta)\, d\theta$$

$$= -84\int_{\theta=\pi}^{\theta=0} 2\sin^2\theta\, d\theta$$

$$= 168\int_{\theta=0}^{\theta=\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= 84\left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=\pi}$$

$$= 84\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - 0\right]$$



$$\int_V \operatorname{div}\bar{F}\, dv = 84\pi$$

To evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$:-

The given surface of the cylinder can be divided into 3 parts.

(i) S_1 : the circular surface $z=0$.

(ii) S_2 : the surface $z=3$ (circular) and

(iii) S_3 : the cylindrical portion of S : $x^2+y^2=4$, $z=0$, $z=3$.

We now find $\iint \bar{F}\cdot\bar{n}\, ds$ over S_1, S_2, S_3.

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iint_{S_1} \bar{F}\cdot\bar{n}\, ds + \iint_{S_2} \bar{F}\cdot\bar{n}\, ds + \iint_{S_3} \bar{F}\cdot\bar{n}\, ds \quad \text{—— (2)}.$$

Case (i) :- To evaluate $\iint_{S_1} F.\bar{n}\,ds$. (OR) The circular surface $z=0$.

Unit normal vector to surface S_1, $z=0$ is $\bar{n}=-\bar{k}$

$$F = 4xi - 2y^2 j + z^2 k .$$

$$\bar{F} = 4xi - 2y^2 j \quad [\because z=0]$$

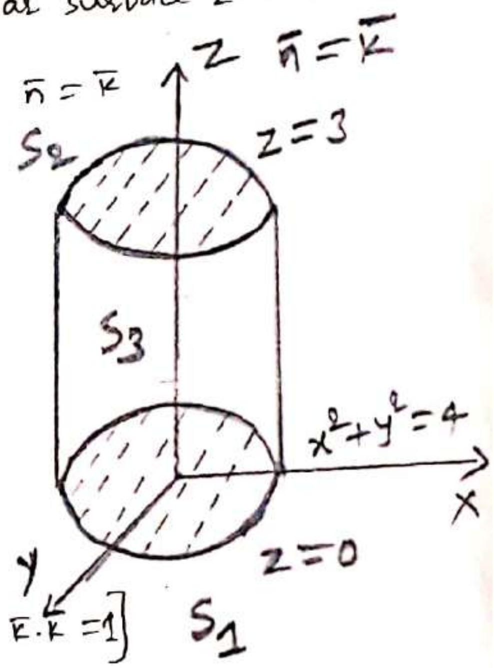
$$\bar{F}.\bar{n} = [4xi - 2y^2 j].(-\bar{k})$$

$$\bar{F}.\bar{n} = 0 .$$

$$\therefore \iint_{S_1} \bar{F}.\bar{n}\,ds = 0 \quad \text{---} \; (3).$$

Case (ii) :- To evaluate $\iint_{S_2} F.\bar{n}\,ds$ (OR) The circular surface $z=3$.

Unit normal vector to surface S_2, $z=3$ is $\bar{n}=\bar{k}$



$$\bar{F} = 4xi - 2y^2 j + z^2 k$$

$$\bar{F} = 4xi - 2y^2 j + 9k \quad [\because z=3]$$

$$\bar{F}.\bar{n} = [4xi - 2y^2 j + 9k].k$$

$$\bar{F}.\bar{n} = 9.$$

$$\iint_{S_2} \bar{F}.\bar{n}\,ds = \iint_R \bar{F}.\bar{n}\,\frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

$$= \iint_R 9.\,\frac{dx\,dy}{1} \quad [\because \bar{n}.\bar{k} = \bar{k}.\bar{k} = 1]$$

$$= 9 \iint_R dx\,dy$$

$$= 9\,[\text{Area of the circle } x^2+y^2=4]$$

$$= 9.\,\pi(2)^2$$

$$\iint_{S_2} \bar{F}.\bar{n}\,ds = 36\pi \quad \text{---} \; (4)$$

Case (iii) To evaluate $\iint_{S_3} \bar{F}.\bar{n}\,ds$ (OR) The cylindrical portion of S: $x^2+y^2=4$, $z=0, z=3$.

Let $\phi = x^2+y^2-4$.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 2(xi + yj)$$

$$|\nabla\phi| = 2\sqrt{x^2+y^2}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj)}{2\sqrt{x^2+y^2}}$$

$$\hat{n} = \frac{xi+yj}{2} \quad \left[\because x^2+y^2=4\right]$$

$$\bar{F}\cdot\hat{n} = \left[4xi - 2y^2 j + z^2 k\right]\cdot\left(\frac{xi+yj}{2}\right)$$

$$= \frac{4x^2-2y^3}{2}$$

$$\bar{F}\cdot\hat{n} = 2x^2-y^3.$$

To evaluate $\iint\limits_{S_3} \bar{F}\cdot\hat{n}\,ds = \iint\limits_{S_3} (2x^2-y^3)\,ds$.

Put $x = 2\cos\theta \quad y = 2\sin\theta$

$ds = 2\,d\theta\,dz$

Here z varies from 0 to 3

$\therefore$ z limits are $z=0 \quad z=3$

θ limits are $\theta=0,\ \theta=2\pi$.

$$\iint\limits_{S_3} \bar{F}\cdot\hat{n}\,ds = \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=3} (8\cos^2\theta - 8\sin^3\theta)\,2\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{\theta=2\pi} (\cos^2\theta - \sin^3\theta)\left[z\right]_{z=0}^{z=3} d\theta.$$

$$= 48\int_{\theta=0}^{\theta=2\pi} \left[\cos^2\theta - \sin^3\theta\right] d\theta$$

$$= 48\int_{\theta=0}^{\theta=2\pi} \left[\frac{1+\cos 2\theta}{2} + \frac{\sin 3\theta}{4} - \frac{3\sin\theta}{4}\right] d\theta$$

$$= 48\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{\cos 3\theta}{12} + \frac{3\cos\theta}{4}\right]_{\theta=0}^{\theta=2\pi}$$

$$= 48\pi \quad \text{——} \quad (5)$$

sub (3) (4) and (5) in (2), we get $\iint\limits_{S} \bar{F}\cdot\hat{n}\,ds = 0 + 36\pi + 48\pi = 84\pi$

$$\therefore \int\limits_{V} \operatorname{div}\bar{F}\,dv = \iint\limits_{S} \bar{F}\cdot\hat{n}\,ds$$

$\therefore$ Gauss Divergence theorem verified.

→ Evaluate $\int_S (yz\,i + xz\,j + xy\,k) \cdot ds$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in the first octant.

Sol:- Given that $\int_S (yz\,i + xz\,j + xy\,k) \cdot ds$. Where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in the first octant.

The surface of the region V: OABC is piecewise smooth comprised of four surfaces.

(i) S_1 : circular quadrant OBC in the yz-plane.

(ii) S_2 : circular quadrant OCA in the zx-plane.

(iii) S_3 : circlcular quadrant OAB in the xy-plane.

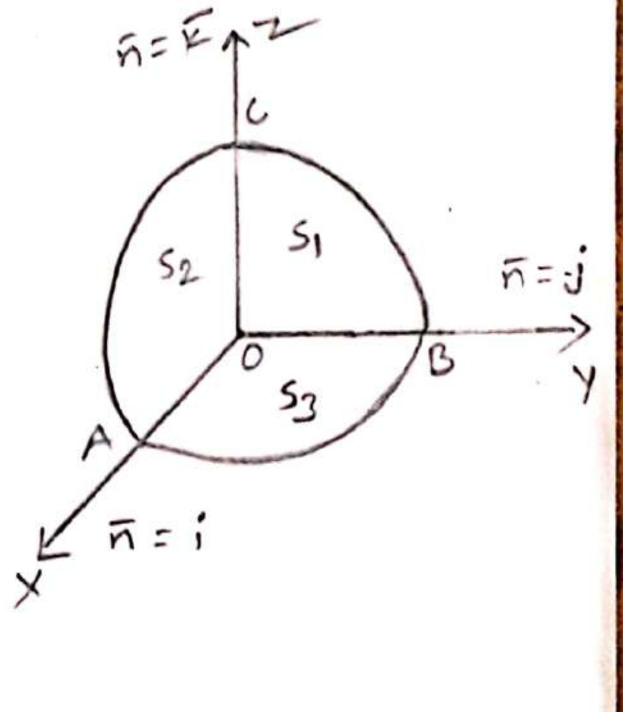
(iv) S : Surface ABC of the sphere in the first octant.

Let $\bar{F} = yz\,i + xz\,j + xy\,k$.

WKT Gauss Divergence theorem.

$$\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds \quad \text{—— (1)}$$

We can write $\int_V \operatorname{div}\bar{F}\, dv = \int_{S_1} \bar{F}\cdot\bar{n}\, ds + \int_{S_2} \bar{F}\cdot\bar{n}\, ds + \int_{S_3} \bar{F}\cdot\bar{n}\, ds + \int_S \bar{F}\cdot\bar{n}\, ds$.



We have $\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\bar{F} = yz\,i + zx\,j + xy\,k$

$F_1 = yz \quad F_2 = zx \quad F_3 = xy$

$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(yz) = 0, \quad \frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(zx) = 0$

$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(xy) = 0$

$\operatorname{div}\bar{F} = 0.$

$\int_V \operatorname{div}\bar{F}\, dv = 0$ —— (2).

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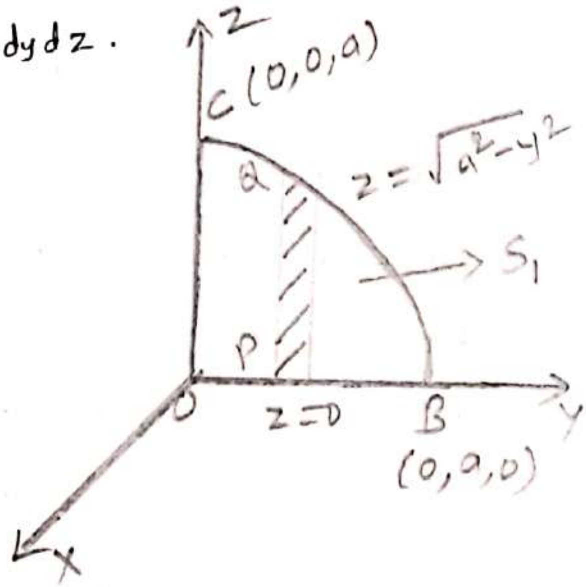
Case (ii):- To evaluate $\int_{S_1}\bar{F}\cdot\bar{n}\,ds$ (OR) on the surface S_1 :-

unit normal vector to the surface S_1 is $\bar{n} = -i$

The surface S_1 is in yz plane., $ds = dy\,dz$.



$$\int_{S_1}\bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|n\cdot i|}$$

$$\bar{F}\cdot\bar{n} = [yz\,i + zx\,j + xy\,k]\cdot(-i)$$

$$\bar{F}\cdot\bar{n} = -yz$$

$$n\cdot i = -i\cdot i = -1$$

$$|n\cdot i| = 1.$$

Here y varies from 0 to a.

$\therefore$ y limits $y=0$, $y=a$.

For each y, z varies from $z=0$ to $\sqrt{a^2-y^2}$

$\therefore$ z limits $z=0$, $z=\sqrt{a^2-y^2}$

$$\int_{S_1}\bar{F}\cdot\bar{n}\,ds = \iint_R -yz\cdot\frac{dy\,dz}{1}$$

$$= -\int_{y=0}^{y=a}\int_{z=0}^{z=\sqrt{a^2-y^2}} yz\,dz\,dy$$

$$= -\int_{y=0}^{y=a} y\cdot\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{a^2-y^2}} dy$$

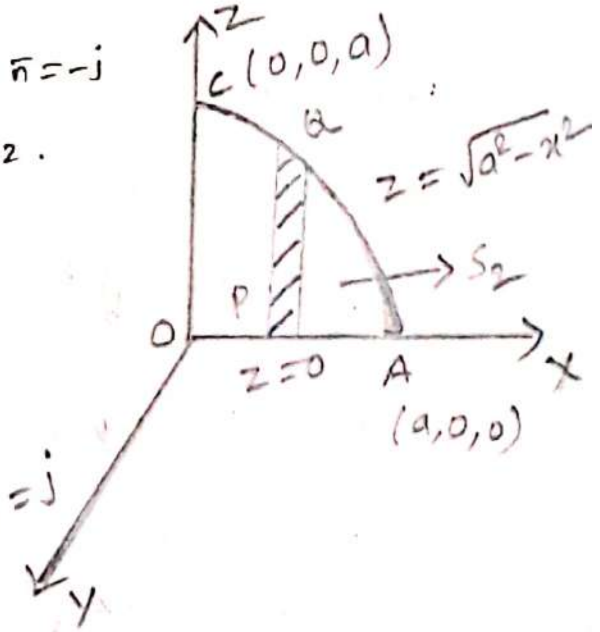
$$= -\frac{1}{2}\int_{y=0}^{y=a} y(a^2-y^2)\,dy$$

$$= -\frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_{y=0}^{y=a}$$

$$= -\frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - 0\right]$$

$$\int_{S_1}\bar{F}\cdot\bar{n}\,ds = -\frac{a^4}{8} \quad \text{——— (3)}$$

Case(ii) To evaluate $\int\int_{S_2} \bar{F}\cdot\bar{n}\,ds$ (OR) on the surface S_2 :-



Unitnormal vector to the surface S_2 is $\bar{n}=-j$

The surface S_2 is in xz plane $ds=dx\,dz$.

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dz}{|n\cdot j|}$$

$$\bar{F}\cdot\bar{n} = [yz\,i + zx\,j + xy\,k]\cdot(-j)$$

$$\bar{F}\cdot\bar{n} = -zx.$$

$$n\cdot j = -j\cdot j = -1$$

$$|n\cdot j| = 1$$

Here x varies from 0 to a.

$\therefore$ x limits $x=0$, $x=a$.

For each x, z varies from 0 to $\sqrt{a^2-x^2}$.

$\therefore$ z limits $z=0$, $z=\sqrt{a^2-x^2}$.

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = \iint_R -zx\,\frac{dx\,dz}{1}$$

$$= -\int_{x=0}^{x=a}\int_{z=0}^{z=\sqrt{a^2-x^2}} xz\,dz\,dx$$

$$= -\int_{x=0}^{x=a} x\cdot\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{a^2-x^2}} dx$$

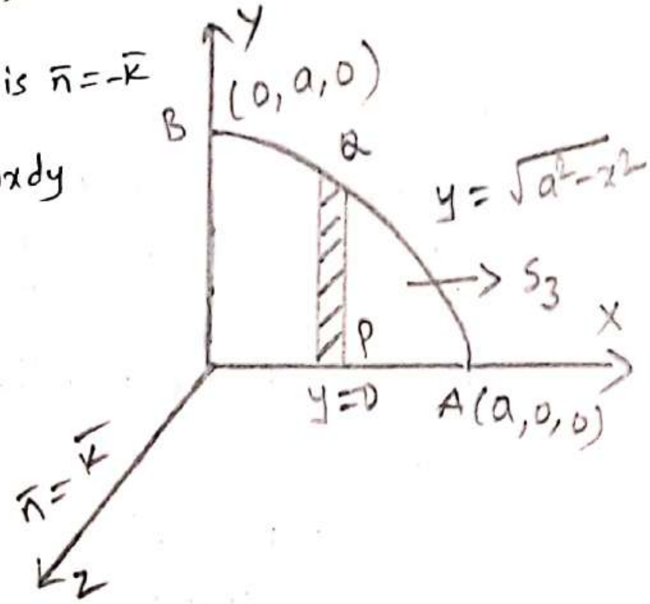
$$= \frac{-1}{2}\int_{x=0}^{x=a} x(a^2-x^2)\,dx.$$

$$= \frac{-1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{x=a}$$

$$= \frac{-1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - 0\right]$$

$$\int_{S_2} \bar{F}\cdot\bar{n}\,ds = -\frac{a^4}{8} \quad \text{——} \quad (4)$$

Case(iii) To evaluate $\int_{S_3} \bar{F}\cdot\bar{n}\, ds$ (OR) On the surface S_3 :-



Unit normal vector to the surface S_3 is $\bar{n} = -\bar{k}$

The surface S_3 is in xy plane, $ds = dx\,dy$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n}\, \frac{dx\,dy}{|\bar{n}\cdot k|}$$

$$\bar{F}\cdot\bar{n} = [yz\,i + xz\,j + xy\,k]\cdot(-\bar{k})$$

$$\bar{F}\cdot\bar{n} = -xy$$

$$\bar{n}\cdot k = -k\cdot k = -1$$

$$|\bar{n}\cdot k| = 1$$

Here x varies from 0 to a

$\therefore$ x limits $x = 0,\ x = a$.

For each x, y varies from $y = 0$ to $\sqrt{a^2-x^2}$

$\therefore$ y limits $y = 0,\ y = \sqrt{a^2-x^2}$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = \iint_R -xy\,.\ \frac{dx\,dy}{1}$$

$$= -\int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} xy\, dy\, dx$$

$$= -\int_{x=0}^{x=a} x\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx$$

$$= -\frac{1}{2}\int_{x=0}^{x=a} x(a^2-x^2)\, dx$$

$$= -\frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{x=a}$$

$$= -\frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - 0\right]$$

$$\int_{S_3} \bar{F}\cdot\bar{n}\, ds = -\frac{a^4}{8} \quad \text{—— (5)}$$

sub. (2) (3) (4) and (5) in (1), we get

$$0 = -\frac{a^4}{8} - \frac{a^4}{8} - \frac{a^4}{8} + \int_S \bar{F}\cdot\bar{n}\,ds \;.$$

$$\therefore \int_S \bar{F}\cdot\bar{n}\,ds = \frac{3a^4}{8} \,.$$

→ Use Divergence theorem to evaluate $\iint (xi+yj+z^2k)\cdot\bar{n}\,ds$ where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z=4$.

sol:- Given that $\iint (xi+yj+z^2k)\cdot\bar{n}\,ds$ where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z=4$.

Let $\bar{F} = xi+yj+z^2k$. , $F = F_1 i + F_2 j + F_3 k$.

W.K.T Gauss Divergence theorem, we have.

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$$

$$\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2).$$

$$\operatorname{div}\bar{F} = 2(1+z)$$

On the cone, $x^2+y^2=z^2$ and $z=4 \Rightarrow x^2+y^2=16$.

The limits are $z=0$ to 4, $y=0$ to $\sqrt{16-x^2}$, $x=0$, to $x=4$.

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv = \int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{16-x^2}}\int_{z=0}^{z=4} 2(1+z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{16-x^2}} 2\left[z+\frac{z^2}{2}\right]_{z=0}^{z=4} dy\,dx.$$

$$= 24\int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{16-x^2}} dy\,dx = 24\int_{x=0}^{x=4}\left[y\right]_{y=0}^{y=\sqrt{16-x^2}} dx.$$

$$= 24\int_{x=0}^{x=4}\sqrt{16-x^2}\,dx.$$

$$= 24\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{x=0}^{x=4}$$

$$= 24\left[0+8\sin^{-1}(1)-0-0\right]$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = 96\pi$$